

# THE QUATERNIONIC GEOMETRY OF 4D CONFORMAL FIELD THEORY

by

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## Abstract

We show that 4-dimensional conformal field theory is most naturally formulated on Kulkarni 4-folds, i. e. real 4-folds endowed with an integrable quaternionic structure. This leads to a formalism that parallels very closely that of 2-dimensional conformal field theory on Riemann surfaces. In this framework, the notion of Fueter analyticity, the quaternionic analogue of complex analyticity, plays an essential role. Conformal fields appear as sections of appropriate either harmonic real or Fueter holomorphic quaternionic line bundles. In the free case, the field equations are statements of either harmonicity or Fueter holomorphicity of the relevant conformal fields. We obtain compact quaternionic expressions of such basic objects as the energy-momentum tensor and the gauge currents for some basic models in terms of Kulkarni geometry. We also find a concise expression of the conformal anomaly and a quaternionic 4-dimensional analogue of the Schwarzian derivative describing the covariance of the quantum energy-momentum tensor. Finally, we analyse the operator product expansions of free fields.

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## 0. Introduction

The success of 2-dimensional conformal field theory both in the study of critical 2-dimensional statistical mechanics and perturbative string theory is well known [1–3]. Higher dimensional conformal field theory is similarly relevant in critical higher dimensional statistical physics and may eventually play an important role in membrane theory [4–6]. Unfortunately, so far it has failed to be as fruitful as its 2-dimensional counterpart in spite of its considerable physical interest.

The basic reason of this failure is well-known. In 2 dimensions, the conformal algebra is infinite dimensional and thus it strongly constraints the underlying field theory. It is precisely this that renders 2-dimensional conformal field theory very predictive and computationally efficient. In  $d > 2$  dimensions, the conformal algebra is instead only  $(d+1)(d+2)/2$  dimensional and has therefore limited structural implications. There are however other features of 2-dimensional conformal field theory, which turn out to be of considerable salience and may generalize to higher dimensions.

In a 2-dimensional conformal model on an oriented Riemann surface  $\Sigma$ , the scale of the background metric in the action can be absorbed into a multiplicative redefinition of the dynamical fields by an appropriate power of the scale. The action can then be expressed entirely in terms of the underlying conformal geometry. The fields become either functions or sections of certain holomorphic line bundles on  $\Sigma$ . In the free case, the field equations reduce to the condition of either harmonicity or holomorphicity of the fields. Complex analyticity is therefore a distinguished feature of these field theoretic models allowing the utilization of powerful methods of complex analysis such as the Cauchy integral formula and the Laurent expansion theorem.

In a higher dimensional conformal model on a manifold  $X$ , the scale of the background metric in the action can be similarly absorbed into a multiplicative redefinition of the fields by some power of the scale and the action is again expressible entirely in terms of the underlying conformal geometry, as in the 2-dimensional case. One may wonder if there are higher dimensional generalizations of 2-dimensional complex analyticity of the same salience. The present paper aims to show that this is in fact so in 4 dimensions. The form of analyticity relevant to the 4-dimensional case is Fueter’s quaternionic analyticity. This is stronger than real analyticity, as complex analyticity is, and yet is weak enough to be fulfilled by a wide class of functions. It also allows for a straightforward generalization of the main fundamental theorems of complex analysis [7].

By definition, a complex function  $f(z)$  of a complex variable  $z$  is holomorphic if it satisfies the well-known Cauchy–Riemann equations  $\partial_{\bar{z}}f = 0$ . Similarly, a quaternionic

function  $f(q)$  of a quaternionic variable  $q$  is right (left) Fueter holomorphic if it satisfies the right (left) Cauchy–Fueter equation  $f\partial_{\bar{q}R} = 0$  ( $\partial_{\bar{q}L}f = 0$ ) [7], where

$$f\partial_{\bar{q}R} = \frac{1}{4}(\partial_{x0}f + \partial_{xr}fj_r), \quad \partial_{\bar{q}L}f = \frac{1}{4}(\partial_{x0}f + j_r\partial_{xr}f). \quad (0.1)$$

for  $q = x^0 + x^r j_r$  with  $x^0, x^r$  real,  $j_r, r = 1, 2, 3$  being the standard generators of the quaternion field  $\mathbb{H}$ . Here, due to the non commutative nature of  $\mathbb{H}$ , one distinguishes between left and right Fueter analyticity.

We know that Riemann surfaces are the largest class of 2–folds allowing for global notions of complex analyticity. It is therefore natural to look for the largest class of 4–folds on which Fueter analyticity can be similarly globally defined.

The closest 4–dimensional analog of a Riemann surface is a Kulkarni 4–fold. A Kulkarni 4–fold  $X$  is a real 4–fold admitting an atlas of quaternionic coordinates  $q$  transforming as

$$q_\alpha = (a_{\alpha\beta}q_\beta + b_{\alpha\beta})(c_{\alpha\beta}q_\beta + d_{\alpha\beta})^{-1} \quad (0.2)$$

for some constant matrix  $\begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{pmatrix} \in \text{GL}(2, \mathbb{H})$  [8]. As the 2–dimensional projective quaternionic group  $\text{PGL}(2, \mathbb{H})$  is isomorphic to the orientation preserving 4–dimensional conformal group  $\text{SO}_0(5, 1)$ , a Kulkarni 4–fold is just an oriented real 4–fold with a conformal structure, much in the same way as a Riemann surface is an oriented 2–fold with a conformal structure. Note the analogy of the transformations (0.2) with the well–known complex Moebius transformations. In 2 dimensions, Moebius coordinates are just one of infinitely many choices of coordinates compatible with the underlying conformal structure. In 4 dimensions, the quaternionic coordinates  $q$  are conversely the only possible choice [9].

The Fueter operators (0.1) appear naturally in the geometry of Kulkarni 4–folds. One can construct a Fueter complex  $(\Omega^0(X, \zeta_*), \delta)$ , where the  $\zeta_p$  are certain quaternionic line bundles on  $X$  and  $\delta$  is a differential built out of  $\partial_q$  and  $\partial_{\bar{q}}$ , and show its equivalence to the standard de Rham complex  $(\Omega^*(X), d)$ . Exploiting this property, one can show that the spaces of closed (anti)selfdual 2–forms, which are two fundamental invariants of every real 4–fold with an oriented conformal structure, are defined by a condition of right (left) Fueter holomorphicity. Kulkarni 4–folds can be further equipped with a harmonic real line bundle  $\rho$  and, in the spin case, with two right/left Fueter holomorphic quaternionic line bundles  $\varpi^\pm$ . The actions of the flat d’Alembertian  $\square = \star 1 \partial_{\bar{q}} \partial_q$  on real sections of  $\rho$  and of the Fueter operator  $\bar{\partial}_R = \partial_{\bar{q}R} d\bar{q}$  ( $\bar{\partial}_L = d\bar{q} \partial_{\bar{q}L}$ ) on quaternionic sections of  $\varpi^+$  ( $\varpi^-$ ) are therefore globally defined. These line bundles and operators are of considerable salience because of their relation with the conformal d’Alembertian and the Dirac operator,

respectively. All the above indicates that Fueter analyticity is a natural notion of regularity on Kulkarni 4-folds.

The family of Kulkarni 4-fold is very vast. It contains such basic examples as  $S^4$  and  $T^4$  and topologically very complicated 4-folds as the oriented 4-dimensional Clifford-Klein forms  $\Gamma \backslash \mathbb{R}^4$ , and  $\Gamma \backslash B_1(\mathbb{R}^4)$ , the oriented 4-dimensional Hopf manifolds  $\Gamma \backslash (S^1 \times S^3)$  and the flat sphere bundles on a Riemann surface  $B_1(\mathbb{R}^2) \times_G S^2$ . Note that all the above 4-folds, like all oriented Riemann surfaces, are Kleinian manifolds.

A Kulkarni 4-fold  $X$  is naturally endowed with a canonical conformal class of locally conformally flat metrics. These are the natural metrics for  $X$ . The Riemann 2-form, the Ricci 1-form and the Ricci scalar of such metrics and all the objects derived from them have particularly simple compact expressions in terms of the scale of the metric and the underlying Kulkarni structure. Exploiting Fueter calculus, one can also derive the general structure of Einstein locally conformally flat metrics, when they exist. Note once more the analogy with the geometry of Riemann surfaces. However, while, in the case of Riemann surfaces, every metric is automatically locally conformally flat and Einstein, the same is no longer true in the case of Kulkarni 4-folds.

On a Kulkarni 4-fold  $X$  equipped with a compatible locally conformally flat metric, the analogy of the geometry of 4- and 2-dimensional conformal field theory becomes manifest. The fields appear as sections of either  $\rho$  or  $\varpi^\pm$  or derived line bundles and the action can be expressed fully in the language of Kulkarni geometry. For instance, the action of the standard conformal complex boson model with  $s/6$  coupling can be cast as

$$I(\phi, \bar{\phi}_c) = -\frac{8}{\pi^2} \int_X \bar{\phi}_c \square \phi, \quad (0.3)$$

with  $\phi$  a complex section of  $\rho$ . The field equations of  $\phi$  read simply as  $\square \phi = 0$  and thus imply the harmonicity of  $\phi$ . Similarly, the action of the standard massless Dirac fermion model can be cast as

$$\begin{aligned} I(\psi^+, \psi^-, \tilde{\psi}^+, \tilde{\psi}^-) &= \frac{2}{\pi^2} \text{Re} \int_X \left[ \tilde{\psi}^+ \bar{\partial}_R \wedge \star dq \psi^- - \psi^+ \bar{\partial}_R \wedge \star dq \tilde{\psi}^- \right] \\ &= \frac{2}{\pi^2} \text{Re} \int_X \left[ \tilde{\psi}^+ \star dq \wedge \bar{\partial}_L \psi^- - \psi^+ \star dq \wedge \bar{\partial}_L \tilde{\psi}^- \right]. \end{aligned} \quad (0.4)$$

with  $\psi^\pm$  complex Grassman sections of  $\varpi^\pm$ .  $\star$  is similar to the Hodge star, but it depends only on the Kulkarni geometry of  $X$ . The field equations of  $\psi^+$  ( $\psi^-$ ) read as  $\psi^+ \bar{\partial}_R = 0$  ( $\bar{\partial}_L \psi^- = 0$ ) and imply the right (left) Fueter holomorphicity of  $\psi^+$  ( $\psi^-$ ). The energy-momentum tensor and the gauge currents have similarly simple expressions and geometrically clear properties in this formalism.

In the quantum case, the operator product expansions of the quantum fields may be formulated and analyzed exploiting harmonicity and Fueter holomorphicity, in a way very close in spirit to the analogous approach of 2-dimensional conformal field theory. One can further define a quaternionic conformally invariant quantum energy-momentum tensor  $T_e$ . The Ward identity obeyed by this can be expressed in terms of the underlying Kulkarni geometry in the form

$$d \star T_e = 0 \quad (0.5)$$

up to contact terms. Under a coordinate change of the form (0.2),  $T_e$  transforms as

$$T_{e\alpha} = \zeta_{3\alpha\beta} (T_{e\beta} + \varrho_{\alpha\beta}). \quad (0.6)$$

Here,  $\varrho_{\alpha\beta}$  depends only on the underlying conformal geometry. So, the matching relation (0.6) is completely analogous to that of the conformally invariant energy-momentum tensor in 2-dimensional conformal field theory and  $\varrho_{\alpha\beta}$  is a 4-dimensional generalization of the Schwarzian derivative.

The present paper is an attempt at generalizing some of the powerful techniques of 2-dimensional conformal field theory to higher dimensional field theory in a geometric perspective. It is similar in spirit to but quite different in approach from the work of refs. [10–11].

In sections 1, 2 and 3, we provide a detailed account of the quaternionic geometry of Kulkarni 4-folds in a way that parallels as much as possible the standard treatment of the geometry of Riemann surfaces. In sections 4 and 5, we analyze the geometric properties of a 4-dimensional conformal field theory on a Kulkarni 4-fold respectively in the classical and quantum case. In section 6, we provide a brief outlook of future developments

## 1. Quaternionic linear algebra and group theory

In this paper, we argue that the geometry underlying 4-dimensional conformal field theory is quaternionic. In this section, we review briefly basic facts of quaternionic linear algebra and group theory.

The quaternion field  $\mathbb{H}$  is the non commutative field generated over  $\mathbb{R}$  by the symbols 1 and  $j_r$ ,  $r = 1, 2, 3$ , subject to the relation <sup>1</sup>

$$j_r j_s = -\delta_{rs} + \epsilon_{rst} j_t. \quad (1.1)$$

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<sup>1</sup> In this paper, we adopt the following conventions. The early Latin indices  $a$  through  $d$  and middle Latin indices  $i$  through  $m$  take the values 0, 1, 2, 3. The middle Latin indices  $e$  through  $g$  and the late Latin indices  $r$  through  $v$  take the values 1, 2, 3. Sum over repeated indices is understood unless they appear on both sides of the same identity.

Hence, a generic quaternion  $a \in \mathbb{H}$  can be written as

$$a = a^0 + a^r j_r, \quad a^0, a^r \in \mathbb{R}. \quad (1.2)$$

Quaternionic conjugation is defined by

$$\bar{a} = a^0 - a^r j_r. \quad (1.3)$$

The real and imaginary parts of a quaternion  $a \in \mathbb{H}$  are defined, in analogy to the complex case, as

$$\operatorname{Re} a = (1/2)(a + \bar{a}) = a^0, \quad \operatorname{Im} a = (1/2)(a - \bar{a}) = a^r j_r. \quad (1.4)$$

The absolute value of a quaternion  $a \in \mathbb{H}$  is given by

$$|a| = (\bar{a}a)^{\frac{1}{2}} = a^0 a^0 + a^r a^r. \quad (1.5)$$

The space  $\mathbb{H}^n$  can be given the structure of right  $\mathbb{H}$  linear space in natural fashion. Further, it can be equipped with the right sesquilinear scalar product defined by

$$\langle u, v \rangle = \sum_{k=1}^n \bar{u}_k v_k, \quad u, v \in \mathbb{H}. \quad (1.6)$$

The  $n$ -dimensional quaternionic general linear group  $\operatorname{GL}(n, \mathbb{H})$  is the group of invertible  $n$  by  $n$  matrices with entries in  $\mathbb{H}$ . Any  $T \in \operatorname{GL}(n, \mathbb{H})$  defines by left matrix action a right  $\mathbb{H}$  linear operator on  $\mathbb{H}^n$ . The  $n$ -dimensional symplectic group  $\operatorname{Sp}(n)$  is the subgroup of  $\operatorname{GL}(n, \mathbb{H})$  formed by those operators leaving the scalar product (1.6) invariant.

$\mathbb{H}\mathbb{P}^n$ , the  $n$  dimensional quaternionic projective space, is the quotient of  $\mathbb{H}^{n+1} - \{0\}$  by the right multiplicative action of the group  $\mathbb{H}_\times$  of non zero quaternions.

The group  $\operatorname{PGL}(n+1, \mathbb{H})$  is defined as

$$\operatorname{PGL}(n+1, \mathbb{H}) = \operatorname{GL}(n+1, \mathbb{H}) / \mathbb{R}_\times, \quad (1.7)$$

where  $\mathbb{R}_\times$  is embedded in  $\operatorname{GL}(n+1, \mathbb{H})$  as the subgroup  $\mathbb{R}_\times 1_{n+1}$ .  $\operatorname{PGL}(n+1, \mathbb{H})$  acts on  $\mathbb{H}\mathbb{P}^n$  by linear fractional transformations.

The case  $n = 1$  will be of special relevance in what follows.  $\operatorname{GL}(1, \mathbb{H})$  is simply the group of non zero quaternions, i. e.  $\operatorname{GL}(1, \mathbb{H}) \cong \mathbb{H}_\times$ .  $\operatorname{Sp}(1)$  is the group of quaternions of unit absolute value, so  $\operatorname{Sp}(1) \cong \operatorname{SU}(2)$ .

(1.2) defines an isomorphism  $\mathbb{R}^4 \cong \mathbb{H}^1$ . Under such an identification, one has [12]

$$\operatorname{Spin}(4) \cong \operatorname{Sp}(1) \times \operatorname{Sp}(1), \quad (1.8)$$

$$\mathrm{SO}(4) \cong (\mathrm{Sp}(1) \times \mathrm{Sp}(1))/\mathbb{Z}_2, \quad (1.9)$$

where  $\mathbb{Z}_2$  is embedded in  $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$  as  $\{\pm(1_1, 1_1)\}$ .

Similarly,  $S^4 \cong \mathbb{HP}^1$ . The group of orientation preserving conformal transformations of  $S^4$  is the connected component of the identity of  $\mathrm{SO}(5, 1)$ ,  $\mathrm{SO}_0(5, 1)$ . The following fundamental isomorphism holds [8–9]

$$\mathrm{SO}_0(5, 1) \cong \mathrm{PGL}(2, \mathbb{H}). \quad (1.10)$$

Explicitly, the action of  $\mathrm{PGL}(2, \mathbb{H})$  on  $\mathbb{HP}^1$  is given by

$$\begin{aligned} T(a) &= (T_{11}a + T_{12})(T_{21}a + T_{22})^{-1}, & a \in \mathbb{HP}^1 \\ &= (-aT_{21}^{-1} + T_{11}^{-1})^{-1}(aT_{22}^{-1} - T_{12}^{-1}), \end{aligned} \quad (1.11)$$

for  $T \in \mathrm{PGL}(2, \mathbb{H})$ . The above isomorphism fails to hold in  $4n$  dimensions with  $n > 1$ , since in fact  $\mathrm{SO}_0(4n + 1, 1) \not\cong \mathrm{PGL}(n + 1, \mathbb{H})$ . This is why the 4-dimensional case is so special.

## 2. The Kulkarni 4-folds

In this paper, we argue that 4 dimensional conformal field theory is formulated most naturally on a class of 4-folds admitting an integrable quaternionic structure, the Kulkarni 4-folds. In the first part of this section, we discuss the local and global quaternionic geometry of such 4-folds. We define the Fueter complex and show its equivalence to the De Rham complex. In the second part, we introduce the natural differential operators of a Kulkarni 4-fold, the d'Alembertian and the Fueter operators, and show their global definition. In the third and final part, we illustrate several basic examples.

### *Local quaternionic differential geometry of real 4-folds*

Let  $X$  be a real 4-fold. Let  $x$  be a local coordinate of  $X$  of domain  $U$ . The four components  $x^i$ ,  $i = 0, 1, 2, 3$ , of  $x$  can be assembled into a quaternionic coordinate  $q$  of the same domain given by

$$q = x^0 + x^r j_r. \quad (2.1)$$

The coordinate vector fields  $\partial_{x_i}$ ,  $i = 0, 1, 2, 3$ , can be similarly organized into a quaternionic vector field  $\partial_q$  given by

$$\partial_q = \frac{1}{4}(\partial_{x_0} - \partial_{x_r} j_r). \quad (2.2)$$

Also,  $\partial_{\bar{q}} = \overline{\partial_q}$ .  $\partial_q$  is a quaternionic differential operator, called Fueter operator, acting on the space of smooth  $\mathbb{H}$ -valued functions  $f$  on  $U$ . Since the quaternion field is not commutative, one must distinguish a left and a right action of  $\partial_q$ :  $f\partial_{qR} = (1/4)(\partial_{x0}f - \partial_{xr}fj_r)$  and  $\partial_{qL}f = (1/4)(\partial_{x0}f - j_r\partial_{xr}f)$ . If  $f$  is  $\mathbb{R}$ -valued, then  $f\partial_{qR} = \partial_{qL}f \equiv \partial_qf$ .  $f$  is right (left) Fueter holomorphic if  $f\partial_{\bar{q}R} = 0$  ( $\partial_{\bar{q}L}f = 0$ ).

The linearly independent wedge products  $dx^{i_1} \wedge \cdots \wedge dx^{i_p}$  with  $0 \leq i_1 < \cdots < i_p \leq 3$  and  $1 \leq p \leq 4$  can similarly be assembled into a distinguished set of alternate wedge products of the differentials  $dq$  and  $d\bar{q}$ :

$$dq = dx^0 + dx^r j_r, \quad (2.3)$$

$$\begin{aligned} -\frac{1}{2}dq \wedge d\bar{q} &= \left(dx^0 \wedge dx^t + \frac{1}{2}\epsilon_{rst}dx^r \wedge dx^s\right)j_t, \\ +\frac{1}{2}d\bar{q} \wedge dq &= \left(dx^0 \wedge dx^t - \frac{1}{2}\epsilon_{rst}dx^r \wedge dx^s\right)j_t, \end{aligned} \quad (2.4)$$

$$\frac{1}{6}dq \wedge d\bar{q} \wedge dq = dx^1 \wedge dx^2 \wedge dx^3 - \frac{1}{2}\epsilon_{rst}dx^0 \wedge dx^r \wedge dx^s j_t, \quad (2.5)$$

$$-\frac{1}{24}dq \wedge d\bar{q} \wedge dq \wedge d\bar{q} = +\frac{1}{24}d\bar{q} \wedge dq \wedge d\bar{q} \wedge dq = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \quad (2.6)$$

All the other combinations of  $dq$  and  $d\bar{q}$  of the same type can be obtained from these by conjugation. Denoting by  $\star$  the Hodge star operator with respect to the flat metric  $h = dx^i \otimes dx^i$  on  $U$ , one has

$$-\frac{1}{2}dq \wedge d\bar{q} = \star\left(-\frac{1}{2}dq \wedge d\bar{q}\right), \quad \frac{1}{2}d\bar{q} \wedge dq = -\star\left(\frac{1}{2}d\bar{q} \wedge dq\right), \quad (2.7)$$

$$\frac{1}{6}dq \wedge d\bar{q} \wedge dq = \star dq, \quad (2.8)$$

$$-\frac{1}{24}dq \wedge d\bar{q} \wedge dq \wedge d\bar{q} = +\frac{1}{24}d\bar{q} \wedge dq \wedge d\bar{q} \wedge dq = \star 1. \quad (2.9)$$

For any  $p$ -form  $\omega = \frac{1}{p!}\omega_{i_1 \dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$  on  $U$  with  $1 \leq p \leq 4$ , one defines the quaternionic components of  $\omega$  by:

$$\omega_q = \omega(\partial_q) = \frac{1}{4}(\omega_0 - \omega_r j_r), \quad p = 1, \quad (2.10)$$

$$\omega_{\bar{q}q} = \omega(\partial_{\bar{q}}, \partial_q) = -\frac{1}{8}(\omega_{0r} + \frac{1}{2}\epsilon_{rst}\omega_{st})j_r, \quad p = 2, \quad (2.11)$$

$$\omega_{q\bar{q}} = \omega(\partial_q, \partial_{\bar{q}}) = +\frac{1}{8}(\omega_{0r} - \frac{1}{2}\epsilon_{rst}\omega_{st})j_r, \quad p = 2, \quad (2.11)$$

$$\omega_{q\bar{q}q} = \omega(\partial_q, \partial_{\bar{q}}, \partial_q) = -\frac{3}{32}(\omega_{123} + \frac{1}{2}\epsilon_{rst}\omega_{0st}j_r), \quad p = 3, \quad (2.12)$$

$$\omega_{\bar{q}q\bar{q}q} = \omega(\partial_{\bar{q}}, \partial_q, \partial_{\bar{q}}, \partial_q) = -\frac{3}{32}\omega_{0123}, \quad p = 4, \quad (2.13)$$

$$\omega_{q\bar{q}q\bar{q}} = \omega(\partial_q, \partial_{\bar{q}}, \partial_q, \partial_{\bar{q}}) = +\frac{3}{32}\omega_{0123}.$$



The remaining components are  $\omega_{\bar{q}} = \omega(\partial_{\bar{q}})$ ,  $p = 1$ , and  $\omega_{\bar{q}q\bar{q}} = \omega(\partial_{\bar{q}}, \partial_q, \partial_{\bar{q}})$ ,  $p = 3$ , and are obtained by conjugation:  $\omega_{\bar{q}} = \overline{\omega_q}$  and  $\omega_{\bar{q}q\bar{q}} = -\overline{\omega_{q\bar{q}q}}$ . One can express  $\omega$  in terms of its components as follows:

$$\omega = 4\text{Re}(\omega_q dq), \quad p = 1, \quad (2.14)$$

$$\omega = -2\text{Re}(\omega_{\bar{q}q} dq \wedge d\bar{q}) - 2\text{Re}(\omega_{q\bar{q}} d\bar{q} \wedge dq), \quad p = 2, \quad (2.15)$$

$$\omega = -\frac{16}{9}\text{Re}(\omega_{q\bar{q}q} dq \wedge d\bar{q} \wedge dq), \quad p = 3, \quad (2.16)$$

$$\omega = \frac{4}{9}\omega_{\bar{q}q\bar{q}q} dq \wedge d\bar{q} \wedge dq \wedge d\bar{q} = \frac{4}{9}\omega_{q\bar{q}q\bar{q}} d\bar{q} \wedge dq \wedge d\bar{q} \wedge dq, \quad p = 4. \quad (2.17)$$

Note that, when  $p = 2$ ,  $\omega$  is  $\star$ -selfdual ( $\star$ -antiselfdual) if and only if  $\omega_{q\bar{q}} = 0$  ( $\omega_{\bar{q}q} = 0$ ).

From (2.10)–(2.13), for any  $p$ -form  $\omega$  on  $U$  with  $0 \leq p \leq 3$ , one has

$$(d\omega)_q = \omega \partial_q R = \partial_q L \omega, \quad p = 0, \quad (2.18)$$

$$(d\omega)_{\bar{q}q} = \partial_{\bar{q}L} \omega_q - \omega_{\bar{q}} \partial_q R, \quad p = 1, \quad (2.19)$$

$$(d\omega)_{q\bar{q}} = \partial_{qL} \omega_{\bar{q}} - \omega_q \partial_{\bar{q}R},$$

$$(d\omega)_{q\bar{q}q} = \frac{3}{2}\partial_{qL} \omega_{\bar{q}q} + \frac{3}{2}\omega_{q\bar{q}} \partial_q R, \quad p = 2, \quad (2.20)$$

$$(d\omega)_{\bar{q}q\bar{q}q} = 2(\partial_{\bar{q}L} \omega_{q\bar{q}q} - \omega_{\bar{q}q\bar{q}} \partial_q R), \quad p = 3, \quad (2.21)$$

$$(d\omega)_{q\bar{q}q\bar{q}} = 2(\partial_{qL} \omega_{\bar{q}q\bar{q}} - \omega_{q\bar{q}q} \partial_{\bar{q}R}).$$

These identities show the relation between the de Rham differential  $d$  and the Fueter operator  $\partial_q$ .

#### *Kulkarni 4-folds*

The Kulkarni  $4n$ -folds are the real  $4n$ -folds uniformized by  $(\mathbb{H}\mathbb{P}^n, \text{PGL}(n+1, \mathbb{H}))$  [8]. This condition turns out to be very restrictive. We are interested in the case where  $n = 1$ .

A Kulkarni 4-fold  $X$  is a real 4-fold<sup>2</sup> with an atlas  $\{(U_\alpha, q_\alpha)\}$  of quaternionic coordinates such that, for  $U_\alpha \cap U_\beta \neq \emptyset$ , there is  $T_{\alpha\beta} \in \text{PGL}(2, \mathbb{H})$  such that

$$q_\alpha = T_{\alpha\beta}(q_\beta), \quad (2.22)$$

where the right hand side is given by (1.11).

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<sup>2</sup> In this paper, we shall assume, unless otherwise stated, that a manifold has no boundary.

*Global quaternionic differential geometry of Kulkarni 4-folds*

Let  $X$  be a Kulkarni 4-fold. The local quaternionic tensorial structures defined on each patch  $U_\alpha$  of  $X$ , as described above, have very simple covariance properties under the coordinate transformations (2.22), as we shall illustrate next.

For  $U_\alpha \cap U_\beta \neq \emptyset$ , we define the matching functions

$$\eta^+_{\alpha\beta} = -q_\alpha T_{\alpha\beta 21} + T_{\alpha\beta 11}, \quad \eta^-_{\alpha\beta} = T_{\alpha\beta 21} q_\beta + T_{\alpha\beta 22}. \quad (2.23)$$

The  $\eta^\pm_{\alpha\beta}$  are nowhere vanishing on  $U_\alpha \cap U_\beta$  since, as will be shown in a moment, the invertible matching operators of the basic quaternionic tensorial structures are polynomial in such objects.

The matching relation of the vector fields  $\partial_{q_\alpha}$  of eq. (2.2) is

$$\partial_{q_\alpha} = \eta^-_{\alpha\beta} \partial_{q_\beta} (\eta^+_{\alpha\beta})^{-1}. \quad (2.24)$$

*Proof.* By differentiating (2.22) using (1.11), one gets (2.25) below, from which one reads off the identity  $\partial_{x_{\beta i}} x_\alpha^0 + \partial_{x_{\beta i}} x_\alpha^s j_s = \eta^+_{\alpha\beta} (\delta_{0i} + \delta_{ri} j_r) (\eta^-_{\alpha\beta})^{-1}$ . Using this relation and (2.2), it is straightforward to derive (2.24). *QED*

The matching relations of the wedge products (2.3)–(2.6) are

$$dq_\alpha = \eta^+_{\alpha\beta} dq_\beta (\eta^-_{\alpha\beta})^{-1}, \quad (2.25)$$

$$dq_\alpha \wedge d\bar{q}_\alpha = |\eta^+_{\alpha\beta}|^2 |\eta^-_{\alpha\beta}|^{-2} \eta^+_{\alpha\beta} dq_\beta \wedge d\bar{q}_\beta (\eta^+_{\alpha\beta})^{-1}, \quad (2.26)$$

$$d\bar{q}_\alpha \wedge dq_\alpha = |\eta^+_{\alpha\beta}|^2 |\eta^-_{\alpha\beta}|^{-2} \eta^-_{\alpha\beta} d\bar{q}_\beta \wedge dq_\beta (\eta^-_{\alpha\beta})^{-1},$$

$$dq_\alpha \wedge d\bar{q}_\alpha \wedge dq_\alpha = |\eta^+_{\alpha\beta}|^2 |\eta^-_{\alpha\beta}|^{-2} \eta^+_{\alpha\beta} dq_\beta \wedge d\bar{q}_\beta \wedge dq_\beta (\eta^-_{\alpha\beta})^{-1}, \quad (2.27)$$

$$dq_\alpha \wedge d\bar{q}_\alpha \wedge dq_\alpha \wedge d\bar{q}_\alpha = |\eta^+_{\alpha\beta}|^4 |\eta^-_{\alpha\beta}|^{-4} dq_\beta \wedge d\bar{q}_\beta \wedge dq_\beta \wedge d\bar{q}_\beta, \quad (2.28)$$

$$d\bar{q}_\alpha \wedge dq_\alpha \wedge d\bar{q}_\alpha \wedge dq_\alpha = |\eta^+_{\alpha\beta}|^4 |\eta^-_{\alpha\beta}|^{-4} d\bar{q}_\beta \wedge dq_\beta \wedge d\bar{q}_\beta \wedge dq_\beta.$$

The Hodge star operators  $\star_\alpha$  associated with the flat metrics  $h_\alpha$  defined above (2.7) match as

$$\star_\alpha = (|\eta^+_{\alpha\beta}| |\eta^-_{\alpha\beta}|^{-1})^{-2(p-2)} \star_\beta \quad \text{on } p\text{-forms.} \quad (2.29)$$

*Proof.* (2.25) follows immediately from differentiating (2.22) using (1.11). (2.26)–(2.28) are trivial consequences of (2.25). (2.29) follows from comparing (2.26)–(2.28) with (2.7)–(2.9). *QED*

The collection  $T = \{T_{\alpha\beta}\}$  associated with the coordinate changes (2.22) defines a flat  $\mathrm{PGL}(2, \mathbb{H})$  1-cocycle on  $X$ . In general, this cocycle cannot be lifted to  $\mathrm{GL}(2, \mathbb{H})$  by choosing suitable  $\mathrm{GL}(2, \mathbb{H})$  representatives of the  $T_{\alpha\beta} \in \mathrm{PGL}(2, \mathbb{H})$ . One has instead a relation of the form

$$T_{\alpha\gamma} = w_{\alpha\beta\gamma} T_{\alpha\beta} T_{\beta\gamma}, \quad w_{\alpha\beta\gamma} = \pm 1, \quad (2.30)$$

whenever  $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ , where  $w = \{w_{\alpha\beta\gamma}\}$  is a flat  $\mathbb{Z}_2$  2-cocycle on  $X$ .

*Proof.* Since  $T$  is a flat  $\mathrm{PGL}(2, \mathbb{H})$  1-cocycle on  $X$  and the center of  $\mathrm{PGL}(2, \mathbb{H})$  is  $\mathbb{R}_\times 1_2$ , (2.30) holds with  $w$  a flat  $\mathbb{R}_\times$  2-cocycle on  $X$ , by a standard theorem of obstruction theory. From here, using (2.23), one can show that relation (2.31) below holds. Now, set  $\phi_{\alpha\beta} = |\eta^+_{\alpha\beta} \eta^-_{\alpha\beta}|^{\frac{1}{2}}$ . Now, using the relation  $T_{\alpha\beta} T_{\beta\alpha} = 1_2$ , implied by (2.30), one can show that either  $T_{\alpha\beta 21} \neq 0$  and  $T_{\beta\alpha 21} \neq 0$  or  $T_{\alpha\beta 21} = 0$  and  $T_{\beta\alpha 21} = 0$  and, using (2.23), one can further verify that  $\phi_{\alpha\beta} = (|T_{\alpha\beta 21}| |T_{\beta\alpha 21}|^{-1})^{\frac{1}{2}}$  in the former case and  $\phi_{\alpha\beta} = (|T_{\alpha\beta 11}| |T_{\beta\alpha 22}|^{-1})^{\frac{1}{2}}$  in the latter case.  $\phi_{\alpha\beta}$  is thus a positive constant and, from its definition, it is clear that  $\phi_{\alpha\gamma} = |w_{\alpha\beta\gamma}| \phi_{\alpha\beta} \phi_{\beta\gamma}$  whenever defined. Hence,  $|w| = \{|w_{\alpha\beta\gamma}|\}$  is a trivial flat  $\mathbb{R}_+$  2-cocycle on  $X$ . So, choices can be made so that  $w$  is a  $\mathbb{Z}_2$  2-cocycle on  $X$ . *QED*

From (2.23) and (2.30), it follows that

$$\eta^\pm_{\alpha\gamma} = w_{\alpha\beta\gamma} \eta^\pm_{\alpha\beta} \eta^\pm_{\beta\gamma}, \quad (2.31)$$

when  $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ . So,  $w$  is the obstruction preventing the smooth  $\mathrm{GL}(1, \mathbb{H})$  1-cochain  $\eta^\pm = \{\eta^\pm_{\alpha\beta}\}$  on  $X$  from being a 1-cocycle.

Note that  $|\eta^\pm| = \{|\eta^\pm_{\alpha\beta}|\}$  is in any case a smooth  $\mathbb{R}_+$  1-cocycle.

The flat  $\mathrm{GL}(2, \mathbb{H})$  1-cochain  $T$  is defined up to a redefinition of the form  $T_{\alpha\beta} \rightarrow c_{\alpha\beta} T_{\alpha\beta}$ , where  $c = \{c_{\alpha\beta}\}$  is a flat  $\mathbb{R}_\times$  1-cocycle. Correspondingly, the smooth  $\mathrm{GL}(1, \mathbb{H})$  1-cochain  $\eta^\pm$  gets redefined as  $\eta^\pm_{\alpha\beta} \rightarrow c_{\alpha\beta} \eta^\pm_{\alpha\beta}$ . Now, the flat  $\mathbb{R}_\times$  1-cocycle  $c$  can be viewed canonically as a pair  $(n, a)$ , where  $n$  and  $a$  are respectively a flat  $\mathbb{R}_+$  1-cocycle and a flat  $\mathbb{Z}_2$  1-cocycle. The geometric structures, which we shall construct below, are independent from  $n$  but do depend on  $a$  in general.

Define

$$\zeta_1 = \eta^-_L \otimes \eta^+_R, \quad (2.32)$$

$$\zeta_2^\pm = |\eta^+|^{-2} \otimes |\eta^-|^2 \otimes \eta^\pm_L \otimes \eta^\pm_R|_{\mathrm{Im} \mathbb{H}}, \quad (2.33)$$

$$\zeta_3 = |\eta^+|^{-2} \otimes |\eta^-|^2 \otimes \eta^-_L \otimes \eta^+_R, \quad (2.34)$$

$$\zeta_4 = |\eta^+|^{-4} \otimes |\eta^-|^4, \quad (2.35)$$

where for  $u, v \in \mathbb{H}_\times \cong \text{GL}(1, \mathbb{H})$ ,  $u_L \otimes v_R$  is the  $\mathbb{R}$  linear operator on  $\mathbb{H}$  defined by  $(u_L \otimes v_R)a = uav^{-1}$  for  $a \in \mathbb{H}$ . Then,  $\zeta_1$  and  $\zeta_3$  are smooth  $(\text{GL}(1, \mathbb{H}) \times \text{GL}(1, \mathbb{H}))/\mathbb{R}_\times$  1-cocycles, where  $\mathbb{R}_\times$  is embedded into  $\text{GL}(1, \mathbb{H}) \times \text{GL}(1, \mathbb{H})$  as  $\mathbb{R}_\times(1_1, 1_1)$ ;  $\zeta_2^\pm$  is a smooth  $\text{PGL}(1, \mathbb{H})$  1-cocycle;  $\zeta_4$  is a smooth  $\mathbb{R}_+$  1-cocycle.

*Proof.* This follows readily from the definitions and from (2.31). QED

Let  $\omega \in \Omega^p(X)$  be a  $p$ -form<sup>3</sup>. Using (2.10)–(2.13), we can associate with  $\omega$  the collection of its local components on the coordinate patches  $U_\alpha$ . If  $\omega \in \Omega^1(X)$ ,  $\omega_q = \{\omega_{q\alpha}\} \in \Omega^0(X, \zeta_1)$  and the map  $\omega \rightarrow \omega_q$  is an  $\mathbb{R}$ -linear isomorphism of  $\Omega^1(X)$  onto  $\Omega^0(X, \zeta_1)$ . On account of (2.29), the spaces  $\Omega^{2\pm}(X)$  of  $\star$ -(anti)selfdual 2-forms on  $X$  are covariantly defined. If  $\omega \in \Omega^{2+}(X)$ ,  $\omega_{\bar{q}q} = \{\omega_{\bar{q}q\alpha}\} \in \Omega^0(X, \zeta_2^+)$  and the map  $\omega \rightarrow \omega_{\bar{q}q}$  is an  $\mathbb{R}$ -linear isomorphism of  $\Omega^{2+}(X)$  onto  $\Omega^0(X, \zeta_2^+)$  and, similarly, if  $\omega \in \Omega^{2-}(X)$ ,  $\omega_{q\bar{q}} = \{\omega_{q\bar{q}\alpha}\} \in \Omega^0(X, \zeta_2^-)$  and the map  $\omega \rightarrow \omega_{q\bar{q}}$  is an  $\mathbb{R}$ -linear isomorphism of  $\Omega^{2-}(X)$  onto  $\Omega^0(X, \zeta_2^-)$ . If  $\omega \in \Omega^3(X)$ ,  $\omega_{q\bar{q}q} = \{\omega_{q\bar{q}q\alpha}\} \in \Omega^0(X, \zeta_3)$  and the map  $\omega \rightarrow \omega_{q\bar{q}q}$  is an  $\mathbb{R}$ -linear isomorphism of  $\Omega^3(X)$  onto  $\Omega^0(X, \zeta_3)$ . Finally, if  $\omega \in \Omega^4(X)$ ,  $\omega_{\bar{q}q\bar{q}q} = \{\omega_{\bar{q}q\bar{q}q\alpha}\} \in \Omega^0(X, \zeta_4)$  and  $\omega_{q\bar{q}q\bar{q}} = \{\omega_{q\bar{q}q\bar{q}\alpha}\} \in \Omega^0(X, \zeta_4)$  and the maps  $\omega \rightarrow \omega_{\bar{q}q\bar{q}q}$  and  $\omega \rightarrow \omega_{q\bar{q}q\bar{q}}$  are both  $\mathbb{R}$ -linear isomorphisms of  $\Omega^4(X)$  onto  $\Omega^0(X, \zeta_4)$ .

*Proof.* This follows easily from the definition of the quaternionic components of the form  $\omega$ , given in (2.10)–(2.13), and from (2.24) upon taking (2.32)–(2.35) into account. For  $p = 1$ , one has  $\omega_{q\alpha} = \omega(\partial_{q\alpha}) = \omega(\eta^{-\alpha\beta}\partial_{q\beta}(\eta^{+\alpha\beta})^{-1}) = \eta^{-\alpha\beta}\omega(\partial_{q\beta})(\eta^{+\alpha\beta})^{-1} = \zeta_{1\alpha\beta}\omega_{q\beta}$ . The proof for the other  $p$  values is analogous. QED

By the above isomorphisms, the standard de Rham complex

$$\begin{array}{ccccccc} & & & d^+ & \Omega^{2+}(X) & & d \\ & & & \nearrow & & \searrow & \\ \Omega^0(X) & \xrightarrow{d} & \Omega^1(X) & & & & \Omega^3(X) \xrightarrow{d} \Omega^4(X) \\ & & & \searrow & & \nearrow & \\ & & & d^- & \Omega^{2-}(X) & & d \end{array} \quad (2.36)$$

is equivalent to the Fueter complex

$$\begin{array}{ccccccc} & & & \delta^+ & \Omega^0(X, \zeta_2^+) & & \delta \\ & & & \nearrow & & \searrow & \\ \Omega^0(X) & \xrightarrow{\delta} & \Omega^0(X, \zeta_1) & & & & \Omega^0(X, \zeta_3) \xrightarrow{\delta} \Omega^0(X, \zeta_4), \\ & & & \searrow & & \nearrow & \\ & & & \delta^- & \Omega^0(X, \zeta_2^-) & & \delta \end{array} \quad (2.37)$$

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<sup>3</sup> Let  $V$  be a  $\mathbb{F}$  vector space. When  $\xi$  is a smooth  $\text{GL}(V)$  1-cocycle on the non empty open subset  $O$  of  $X$ , we denote by  $\Omega^p(O, \xi)$  the  $\mathbb{F}$  vector space of  $p$ -form sections of  $\xi$  on  $O$ . In particular,  $\Omega^p(O)$  is the space of real  $p$ -forms on  $O$ .

where the Fueter operators  $\delta$  are defined by the right hand sides of (2.18)–(2.21), with  $\delta^+$  and  $\delta^-$  corresponding respectively to the first and second expression (2.19). The two definitions of the last  $\delta$  differ only by their sign.

From here, one sees that a 2-form  $\omega \in \Omega^{2+}(X)$  ( $\omega \in \Omega^{2-}(X)$ ) is closed if and only if  $\omega_{\bar{q}q} \partial_{\bar{q}R} = 0$  ( $\partial_{\bar{q}L} \omega_{q\bar{q}} = 0$ ), that is if and only if  $\omega_{\bar{q}q}$  ( $\omega_{q\bar{q}}$ ) is right (left) Fueter holomorphic.

*Proof.* Let  $\omega \in \Omega^{2+}(X)$ . Then,  $\omega_{q\bar{q}} = 0$ . So, if further  $d\omega = 0$ , one has  $\omega_{\bar{q}q} \partial_{\bar{q}R} = -\overline{\partial_{qL} \omega_{\bar{q}q}} = -\frac{2}{3} \overline{(d\omega)_{q\bar{q}q}} = 0$ , by (2.11) and (2.20). The corresponding statement for a closed  $\omega \in \Omega^{2-}(X)$  can be proven in analogous manner. *QED*

The spaces of closed (anti)selfdual 2-forms are important invariants of any real 4-fold endowed with a conformal structure. The above proposition shows that, on a Kulkarni 4-fold, such spaces are defined by a condition of Fueter holomorphicity. We believe that this result highlights quite clearly the relevance of Fueter analyticity to the geometry of Kulkarni 4-folds.

*The 1-cocycle  $\rho$  and the d'Alembert operator  $\square$*  <sup>4</sup>

Let  $X$  be a Kulkarni 4-fold. We set

$$\rho = |\eta^+|^{-1} \otimes |\eta^-|. \quad (2.38)$$

Then,  $\rho$  is a smooth  $\mathbb{R}_+$  1-cocycle.

*Proof.* This follows immediately from the definition and from (2.31). *QED*

Let  $F \in \Omega^0(X, \rho)$ . Set

$$\square F = \partial_{\bar{q}} \partial_q F \star 1 \quad (2.39)$$

on each coordinate patch. Then,  $\square F = \{(\square F)_\alpha\} \in \Omega^4(X, \rho^{-1})$ .

*Proof.* Let  $(U, q)$  be a quaternionic chart of  $X$  and let  $f \in \Omega^0(U)$ . Then,

$$\partial_{\bar{q}} \partial_q f \star 1 = \frac{1}{16} d \star df. \quad (2.40)$$

This relation can be easily checked by evaluating the right hand side in terms of the components of the real coordinate  $x$  contained in  $q$  (cf. eq. (2.1)). Using (2.29), (2.40) and (2.38) and the matching relation  $F_\alpha = \rho_{\alpha\beta} F_\beta$ , one finds

$$\partial_{\bar{q}\alpha} \partial_{q\alpha} F_\alpha \star_\alpha 1 = -\partial_{\bar{q}\beta} \partial_{q\beta} (\rho_{\alpha\beta}^{-1}) F_\beta \star_\beta 1 + \rho_{\alpha\beta}^{-1} \partial_{\bar{q}\beta} \partial_{q\beta} F_\beta \star_\beta 1. \quad (2.41)$$

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<sup>4</sup> In the mathematical literature, this operator is usually called Laplacian and is denoted by  $\Delta$ . In the spirit of field theory, we rather think of it as the Euclidean version of the d'Alembert operator  $\square$ .

Now, using the relation  $T_{\alpha\beta}T_{\beta\alpha} = 1_2$  implied by (2.30), one can show that either  $T_{\alpha\beta 21} \neq 0$  and  $T_{\beta\alpha 21} \neq 0$  or  $T_{\alpha\beta 21} = 0$  and  $T_{\beta\alpha 21} = 0$  and, using further (2.23), one can verify that  $\rho_{\alpha\beta} = |T_{\alpha\beta 21}||T_{\beta\alpha 21}||q_\beta + T_{\alpha\beta 21}^{-1}T_{\alpha\beta 22}|^2$  in the former case and  $\rho_{\alpha\beta} = |T_{\alpha\beta 22}||T_{\beta\alpha 11}|$  in the latter case. Using these expressions, one finds that

$$\partial_{\bar{q}\beta}\partial_{q\beta}(\rho_{\alpha\beta}^{-1}) = 0 \quad (2.42)$$

by direct computation. The statement follows now readily from (2.41) and (2.42). *QED*

Note that, in terms of the real coordinate  $x$  contained in  $q$ ,  $\square F = \frac{1}{16}(\partial_{x0}\partial_{x0} + \partial_{xr}\partial_{xr})F \star 1$ . So,  $\square$  is essentially the euclidean d'Alembertian operator. (2.42) shows then that the 1-cocycle  $\rho$  is harmonic. This allows for a global definition of harmonicity on a Kulkarni 4-fold  $X$ . An element  $F \in \Omega^0(X, \rho)$  is said harmonic if  $\square F = 0$ . In such a case,  $F$  is given locally by the real part of some Fueter holomorphic function  $K$  [7].

*The 1-cocycles  $\varpi^\pm$  and the operators  $\bar{\partial}_{R,L}$*

Let  $X$  be a Kulkarni 4-fold such that  $w = 1$ . We set

$$\varpi^+ = |\eta^+|^{\frac{1}{2}} \otimes |\eta^-|^{-\frac{3}{2}} \otimes \eta^+_{R}, \quad \varpi^- = |\eta^+|^{-\frac{3}{2}} \otimes |\eta^-|^{\frac{1}{2}} \otimes \eta^-_{L}, \quad (2.43)$$

where for  $u \in \mathbb{H}_\times \cong \text{GL}(1, \mathbb{H})$ ,  $u_R$  ( $u_L$ ) is the the left (right)  $\mathbb{H}$  linear operator on  $\mathbb{H}$  defined by  $u_R a = a u^{-1}$  ( $u_L a = u a$ ) for  $a \in \mathbb{H}$ . Then,  $\varpi^\pm$  is a smooth  $\text{GL}(1, \mathbb{H})$  1-cocycle on  $X$ .

*Proof.* This follows readily from (2.31), taking into account that  $w = 1$  in this case, by assumption. *QED*

Note that  $\varpi^\pm$  depends on the choice of a  $\mathbb{Z}_2$  1-cocycle  $a$ , as discussed above (2.32). We assume that a choice is made once and for all.

For  $\Phi \in \Omega^0(X, \varpi^+)$  and  $\Psi \in \Omega^0(X, \varpi^-)$ , we set

$$\Phi \bar{\partial}_R = \Phi \partial_{\bar{q}R} d\bar{q}, \quad \bar{\partial}_L \Psi = d\bar{q} \partial_{\bar{q}L} \Psi \quad (2.44)$$

on each coordinate patch. Then,  $\Phi \bar{\partial}_R = \{(\Phi \bar{\partial}_R)_\alpha\} \in \Omega^1(X, \varpi^+)$  and  $\bar{\partial}_L \Psi = \{(\bar{\partial}_L \Psi)_\alpha\} \in \Omega^1(X, \varpi^-)$ .

*Proof.* We show only that  $\Phi \bar{\partial}_R \in \Omega^1(X, \varpi^+)$ , since the proof of the corresponding statement for  $\Psi$  is totally analogous. For the rest of the proof, introducing a slightly inconsistent notation, we denote by  $\varpi^\pm$  the matching functions defined by (2.43) with the indices  $R, L$  suppressed. Let  $(U, q)$  be a quaternionic chart of  $X$  and let  $f \in \Omega^0(U) \otimes \mathbb{H}$ . Then, one has

$$f \partial_{\bar{q}R} \star 1 = \frac{1}{4} df \wedge \star dq. \quad (2.45)$$

(2.45) can be easily be checked by expressing both sides in terms of the components of the real coordinate  $x$  contained in  $q$ . Using (2.45) and the matching relation  $\Phi_\alpha = \Phi_\beta \varpi^+_{\beta\alpha}$ , one has

$$\Phi_\alpha \partial_{\bar{q}\alpha R} \star_\alpha 1 = \frac{1}{4} d\Phi_\beta \wedge \varpi^+_{\beta\alpha} \star_\alpha dq_\alpha + \Phi_\beta \varpi^+_{\beta\alpha} \partial_{\bar{q}\alpha R} \star_\alpha 1. \quad (2.46)$$

From (2.43) and (2.23), one computes

$$\begin{aligned} \varpi^+_{\beta\alpha} \partial_{\bar{q}\alpha R} &= |\eta^+_{\alpha\beta}|^{-\frac{5}{2}} |\eta^-_{\beta\alpha}|^{-\frac{3}{2}} \left\{ \overline{\eta^+_{\alpha\beta}} \left[ -\frac{5}{4} |\eta^+_{\alpha\beta}|^{-2} (|\eta^+_{\alpha\beta}|^2) \partial_{\bar{q}\alpha R} \right. \right. \\ &\quad \left. \left. - \frac{3}{4} |\eta^-_{\beta\alpha}|^{-2} (|\eta^-_{\beta\alpha}|^2) \partial_{\bar{q}\alpha R} \right] + \overline{\eta^+_{\alpha\beta}} \partial_{\bar{q}\alpha R} \right\} \\ &= -\frac{3}{8} |\eta^+_{\alpha\beta}|^{-\frac{5}{2}} |\eta^-_{\beta\alpha}|^{-\frac{3}{2}} (\eta^-_{\beta\alpha})^{-1} \overline{[T_{\beta\alpha 21} \eta^+_{\alpha\beta} + \eta^-_{\beta\alpha} T_{\alpha\beta 21}]}. \end{aligned} \quad (2.47)$$

Using the relation  $T_{\alpha\beta} T_{\beta\alpha} = 1_2$ , following from (2.30), and (2.23), one finds that

$$T_{\beta\alpha 21} \eta^+_{\alpha\beta} + \eta^-_{\beta\alpha} T_{\alpha\beta 21} = 0. \quad (2.48)$$

Combining (2.47) and (2.48), one concludes that

$$\varpi^+_{\beta\alpha} \partial_{\bar{q}\alpha R} = 0. \quad (2.49)$$

From (2.8), (2.27) and (2.43), one verifies further that

$$\frac{1}{4} \varpi^+_{\beta\alpha} \star_\alpha dq_\alpha = \frac{1}{4} \star_\beta dq_\beta \varpi^-_{\beta\alpha}. \quad (2.50)$$

By (2.46), (2.49) and (2.50), one has, using (2.45),

$$\begin{aligned} \Phi_\alpha \partial_{\bar{q}\alpha R} \star_\alpha 1 &= \frac{1}{4} d\Phi_\beta \wedge \star_\beta dq_\beta \varpi^-_{\beta\alpha} \\ &= \Phi_\beta \partial_{\bar{q}\beta R} \varpi^-_{\beta\alpha} \star_\beta 1. \end{aligned} \quad (2.51)$$

From this relation, using (2.43), (2.25) and (2.29) with  $p = 0$ , it is a simple matter to check that  $(\Phi \bar{\partial}_R)_\alpha = (\Phi \bar{\partial}_R)_\beta \varpi^+_{\beta\alpha}$ , showing the statement. *QED*

(2.49) and its left analog show that the 1-cocycle  $\varpi^+$  ( $\varpi^-$ ) is right (left) Fueter holomorphic. This allows for a global definition of Fueter holomorphicity on a Kulkarni 4-fold  $X$ . An element  $\Phi \in \Omega^0(X, \varpi^+)$  ( $\Psi \in \Omega^0(X, \varpi^-)$ ) is right (left) Fueter holomorphic if  $\Phi \bar{\partial}_R = 0$  ( $\bar{\partial}_L \Psi = 0$ ).

#### *Topological properties of Kulkarni 4-folds*

On account of the isomorphism (1.10), (2.22) entails that a Kulkarni 4-fold is just a real 4-fold with an integrable oriented conformal structure.

A Kulkarni 4-fold structure entails a reduction of the structure group of  $X$  from  $\mathrm{GL}(4, \mathbb{R})$  to  $(\mathrm{GL}(1, \mathbb{H}) \times \mathrm{GL}(1, \mathbb{H}))/\mathbb{R}_\times$ .

*Proof.* Indeed, from (2.24), it appears that the smooth 1-cocycle implementing the matching relations in  $TX$  is the  $(\mathrm{GL}(1, \mathbb{H}) \times \mathrm{GL}(1, \mathbb{H}))/\mathbb{R}_\times$  1-cocycle  $\eta^-_L \otimes \eta^+_R$ . *QED*

The resulting  $(\mathrm{GL}(1, \mathbb{H}) \times \mathrm{GL}(1, \mathbb{H}))/\mathbb{R}_\times$  structure on  $X$ , being yielded by coordinates, is integrable.

Since  $(\mathrm{GL}(1, \mathbb{H}) \times \mathrm{GL}(1, \mathbb{H}))/\mathbb{R}_\times$  is a connected group,  $X$  is oriented. Hence, the first Stieffel–Whitney class of  $X$  vanishes:

$$w_1(X) = 1. \quad (2.52)$$

The flat  $\mathbb{Z}_2$  2-cocycle  $w$  appearing in (2.30) defines a cohomology class  $w \in H^2(X, \mathbb{Z}_2)$ . It can be seen that  $w$  is precisely the second Stieffel–Whitney class of  $X$ :

$$w_2(X) = w. \quad (2.53)$$

*Proof.*  $|\eta^+| \otimes |\eta^-|^{-1}$  is a smooth  $\mathbb{R}_+$  1-cocycle, hence, it is trivial. So, the smooth  $(\mathrm{GL}(1, \mathbb{H}) \times \mathrm{GL}(1, \mathbb{H}))/\mathbb{R}_\times$  1-cocycle  $\eta^-_L \otimes \eta^+_R$  is equivalent to the smooth  $(\mathrm{Sp}(1) \times \mathrm{Sp}(1))/\mathbb{Z}_2$  1-cocycle  $\theta^-_L \otimes \theta^+_R$ , where  $\mathbb{Z}_2$  is embedded in  $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$  as  $\{\pm(1_1, 1_1)\}$  and  $\theta^\pm = |\eta^\pm|^{-1} \otimes \eta^\pm$  is an  $\mathrm{Sp}(1)$  1-cochain. This yields a reduction of the structure group of  $X$  from  $(\mathrm{GL}(1, \mathbb{H}) \times \mathrm{GL}(1, \mathbb{H}))/\mathbb{R}_\times$  to  $(\mathrm{Sp}(1) \times \mathrm{Sp}(1))/\mathbb{Z}_2$ . Now  $\theta^\pm$  satisfies relation (2.31) with  $\eta^\pm$  substituted by  $\theta^\pm$ . From the isomorphisms (1.8) and (1.9), it follows then that the  $\mathbb{Z}_2$  2-cocycle  $w$  is precisely the obstruction to lifting the structure group of  $X$  from  $\mathrm{SO}(4)$  to  $\mathrm{Spin}(4)$ . This identifies  $w$  as a representative of the second Stieffel–Whitney class of  $X$ . *QED*

So, the spin Kulkarni 4-folds are precisely those for which  $w = 1$ . In such a case, the spin structures correspond precisely to the choices of the  $\mathbb{Z}_2$  1-cocycle  $a$  on  $X$  discussed above (2.32). Indeed, as is well-known, such choices describe the cohomology group  $H^1(X, \mathbb{Z}_2)$ .

As  $X$  is endowed with an integrable oriented conformal structure, the first Pontryagin class of  $X$  is zero:

$$p_1(X) = 0. \quad (2.54)$$

*Proof.* The integrability of the conformal structure implies the existence of locally conformally flat metrics, whose Weyl 2-form vanishes [9]. The Pontryagin density, which is quadratic in the components of  $W(g)$ , consequently vanishes too. *QED*



Let  $X$  be compact. As  $p_1(X) = 0$ , the signature of  $X$  vanishes as well,  $\sigma(X) = 0$ . This entails that the Euler characteristic of  $X$  is even:

$$\chi(X) \in 2\mathbb{Z}. \quad (2.55)$$

If  $X$  is compact, then (2.54) and (2.55) imply that  $X$  bounds an oriented 5-fold by the Thom Pontryagin theorem [8].

All the 1-cocycles defined in the previous subsections yield smooth vector bundles on  $X$  in the usual manner. In particular,  $\zeta_1$  and  $\zeta_3$  are smooth  $(\mathrm{GL}(1, \mathbb{H}) \times \mathrm{GL}(1, \mathbb{H}))/\mathbb{R}_\times$  line bundles, the  $\zeta_2^\pm$  are smooth  $\mathrm{PGL}(1, \mathbb{H})$  line bundles,  $\zeta_4$  and  $\rho$  are smooth  $\mathbb{R}_+$  line bundles and the  $\varpi^\pm$  are  $\mathrm{GL}(1, \mathbb{H})$  line bundles.

The operator  $\square$  is elliptic. Therefore, when  $X$  is compact, the subspace of the harmonic  $F \in \Omega^0(X, \rho)$  is finite dimensional. The operators  $\bar{\partial}_{R,L}$  are also elliptic. Hence, if  $X$  is compact, the subspace of right (left) Fueter holomorphic  $\Phi \in \Omega^0(X, \varpi^+)$  ( $\Psi \in \Omega^0(X, \varpi^-)$ ) is similarly finite dimensional. In the next section, we shall show that  $\square$  and the  $\bar{\partial}_{R,L}$  are related respectively to the conformal d'Alembertian and to a certain Dirac operator. This will allow us to derive vanishing theorems.

#### *Kulkarni automorphisms*

An orientation preserving diffeomorphism  $f$  of  $X$  is a Kulkarni automorphism of  $X$  if  $q_\alpha \circ f \circ q_\beta^{-1}$ , whenever defined, is a restriction of some element of  $\mathrm{PGL}(2, \mathbb{H})$ . The Kulkarni automorphisms of  $X$  form a group under composition,  $\mathrm{Aut}(X)$ .

#### *Examples of Kulkarni 4-folds*

The basic example of Kulkarni 4-fold is  $\mathbb{HP}^1$ . As a 4-fold  $\mathbb{HP}^1 \cong S^4$ . Indeed,  $\mathbb{HP}^1$  can be covered by two quaternionic charts  $(q_\alpha, U_\alpha)$ ,  $\alpha = 1, 2$ , where  $U_\alpha = \{(p_1, p_2) \in \mathbb{H}^2 - \{(0, 0)\} | p_\alpha \neq 0\} / \mathbb{H}_\times$  and  $q_1 = p_2 p_1^{-1}$  and  $q_2 = -p_1 p_2^{-1}$ . One has  $q_2 = -(q_1)^{-1}$  on the overlap  $U_1 \cap U_2$ . Under the isomorphism  $\mathbb{H}^1 \cong \mathbb{R}^4$ , this matching relation is equivalent to that of the customary stereographic projection of  $S^4$ . Clearly,  $\mathrm{Aut}(\mathbb{HP}^1) = \mathrm{PGL}(2, \mathbb{H})$ . Also,  $w(\mathbb{HP}^1) = 1$ .

Let  $D$  be a simply connected non empty open subset of  $\mathbb{HP}^1$ . Then,  $D$  is a Kulkarni 4-fold with the Kulkarni structure induced by that of  $\mathbb{HP}^1$ . When  $D$  is a proper subset of  $\mathbb{HP}^1$ , then  $D$  can be covered by a single quaternionic chart  $(q, U)$  with  $U = D$ . The automorphism group  $\mathrm{Aut}(D)$  of  $D$  is the subgroup of  $\mathrm{PGL}(2, \mathbb{H})$  mapping  $D$  onto itself. Clearly,  $w(D) = 1$ .

A Kleinian group  $\Gamma$  for  $D$  is a subgroup of  $\mathrm{Aut}(D)$  acting freely and properly discontinuously on  $D$  [9,13]. The Kleinian manifold  $\Gamma \backslash D$  is then a Kulkarni 4-fold, as it is a real 4-fold uniformized by  $(\mathbb{HP}^1, \mathrm{PGL}(2, \mathbb{H}))$ .  $\mathrm{Aut}(\Gamma \backslash D)$  can be identified with the normalizer of  $\Gamma$  in  $\mathrm{Aut}(D)$ .  $w(\Gamma \backslash D) = 1$  if and only if  $\Gamma$  can be lifted to a subgroup of  $\mathrm{GL}(2, \mathbb{H})$ .

We consider next several standard examples.

*i)*  $D = \mathbb{HP}^1$ .  $\text{Aut}(\mathbb{HP}^1) = \text{PGL}(2, \mathbb{H})$ , as shown earlier. By a simple argument based on Lefschetz's fixed point theorem, it is easy to see that there is no non trivial Kleinian group  $\Gamma$  for  $\mathbb{HP}^1$ , since every  $T \in \text{PGL}(2, \mathbb{H})$  has at least a fixed point in  $\mathbb{HP}^1$ . Thus, there are no Kulkarni 4-folds covered by  $\mathbb{HP}^1$  except for  $\mathbb{HP}^1$  itself.

*ii)*  $D = \mathbb{H}^1$ . It appears that  $\mathbb{H}^1 \cong \mathbb{R}^4$ , as a 4-fold.  $\text{Aut}(\mathbb{H}^1)$  is the subgroup of  $\text{PGL}(2, \mathbb{H})$  formed by those  $T$  such that  $T_{21} = 0$ . There are plenty of Kleinian groups  $\Gamma$  for  $\mathbb{H}^1$ . Among these, the orientation preserving 4-dimensional Bieberbach groups, which have been classified [13]. In this way, the Kulkarni 4-folds  $\Gamma \backslash \mathbb{H}$  covered by  $\mathbb{H}^1$  include the 4-torus  $T^4$  and the oriented 4-folds finitely covered by it.

*iii)*  $D = B_1(\mathbb{H}^1)$ . As a 4-fold,  $B_1(\mathbb{H}^1) \cong B_1(\mathbb{R}^4)$ , the unit ball in  $\mathbb{R}^4$ .  $\text{Aut}(B_1(\mathbb{H}^1))$  is the subgroup of  $\text{PGL}(2, \mathbb{H})$  formed by those  $T$  such that  $|T_{11}|^2 - |T_{21}|^2 = |T_{22}|^2 - |T_{12}|^2 = k$  for some  $k \in \mathbb{R}_+$  and  $\bar{T}_{11}T_{12} - \bar{T}_{21}T_{22} = 0$ . There are plenty of Kleinian groups  $\Gamma$  for  $B_1(\mathbb{H}^1)$ . The Kulkarni 4-folds  $\Gamma \backslash B_1(\mathbb{H}^1)$  covered by  $B_1(\mathbb{H}^1)$  are the 4-dimensional analogue of higher genus Riemann surfaces.

*iv)*  $D = \mathbb{H}^1 - \{0\}$ . As a 4-fold,  $\mathbb{H}^1 - \{0\} \cong \mathbb{R}^4 - \{0\}$ .  $\text{Aut}(\mathbb{H}^1 - \{0\})$  contains as a subgroup of index 2 the subgroup of  $\text{PGL}(2, \mathbb{H})$  formed by those  $T$  such that  $T_{12} = T_{21} = 0$ . There are plenty of Kleinian groups  $\Gamma$  for  $\mathbb{H}^1 - \{0\}$ . Among the Kulkarni 4-folds  $\Gamma \backslash (\mathbb{H}^1 - \{0\})$  covered by  $\mathbb{H}^1 - \{0\}$ , there are the oriented 4-dimensional Hopf manifolds, that is  $S^3 \times S^1$  and the oriented compact 4-folds finitely covered by it.

*v)*  $D = \mathbb{H}^1 - \mathbb{R}^1$ . As a 4-fold,  $\mathbb{H}^1 - \mathbb{R}^1 \cong \mathbb{R}^4 - \mathbb{R}^1$ . There are many Kleinian groups  $\Gamma$  for  $\mathbb{H}^1 - \mathbb{R}^1$ . The Kulkarni 4-folds  $\Gamma \backslash (\mathbb{H}^1 - \mathbb{R}^1)$  covered by  $\mathbb{H}^1 - \mathbb{R}^1$  include the flat  $S^2$  fiber bundle on a compact Riemann surface, as  $\mathbb{R}^4 - \mathbb{R}^1 \cong B_1(\mathbb{R}^2) \times S^2$ .

### 3. The geometry of Kulkarni 4-folds from a Riemannian point of view

4-dimensional conformal field theory is most naturally formulated in a locally conformally flat metric background. One expects calculations to simplify considerably if this background has special properties, such as having a large group of isometries or being Einstein. A Kulkarni 4-fold is equipped with a canonical conformal equivalence class of locally conformally flat metrics. These are studied in the first part of this section using the quaternionic geometric framework introduced above. We also derive conditions for the existence of an Einstein representative in the class and its general form, when it exists. In the second part of the section, we show that the operators  $\square$  and the  $\bar{\partial}_{R,L}$  are related respectively to the conformal d'Alembertian and to a certain Dirac operator. This will allow

us to derive vanishing theorems à la Bochner for their kernels. Examples are provided in the third and final part of the section.

*Local quaternionic Riemannian geometry of a real 4-fold*

Let  $X$  be a real 4-fold. Let  $x$  be a local coordinate of  $X$  of domain  $U$ . On  $U$ , one can define the conformally flat vierbein

$$e_a = e^{-\varphi} \delta_a^i \partial_{x^i}. \quad (3.1)$$

Its dual vierbein is

$$e^\vee_a = e^\varphi \delta_{ai} dx^i. \quad (3.2)$$

The associated metric is

$$g = e^\vee_a \otimes e^\vee_a = e^{2\varphi} dx^i \otimes dx^i. \quad (3.3)$$

The components of the vierbein  $e_a$ ,  $a = 0, 1, 2, 3$ , can be assembled into the quaternionic einbein

$$e = (1/4)(e_0 - e_f j_f). \quad (3.4)$$

By (2.2) and (3.1),  $e$  is given by

$$e = e^{-\varphi} \partial_q. \quad (3.5)$$

Similarly, the components of the dual vierbein  $e^\vee_a$ ,  $a = 0, 1, 2, 3$ , can be assembled into the quaternionic dual einbein

$$e^\vee = e^\vee_0 + e^\vee_f j_f. \quad (3.6)$$

From (2.3) and (3.2), one has

$$e^\vee = e^\varphi dq. \quad (3.7)$$

The metric  $g$  is then given by

$$g = \text{Re}(\bar{e}^\vee \otimes e^\vee) = e^{2\varphi} \text{Re}(d\bar{q} \otimes dq). \quad (3.8)$$

The Hodge star operator  $*$  of  $g$  is related to  $\star$  as

$$* = e^{-2(p-2)\varphi} \star \quad \text{on } p\text{-forms.} \quad (3.9)$$

Many formulae of Riemannian geometry take a particularly compact form when expressed in terms of  $e$  and  $e^\vee$ . Below, we shall adopt the Cartan formulation of Riemannian geometry.

The components of the spin connection  $\omega_{ab}$  1-form can be organized into the two quaternionic 1-forms

$$\begin{aligned}\omega^+ &= -\frac{1}{4}\left(\omega_{00} + \omega_{0f}\bar{j}_f + \omega_{e0}j_e + \omega_{ef}j_e\bar{j}_f\right) = \frac{1}{2}\left(\omega_{0g} + \frac{1}{2}\epsilon_{efg}\omega_{ef}\right)j_g, \\ \omega^- &= +\frac{1}{4}\left(\omega_{00} + \omega_{0f}j_f + \omega_{e0}\bar{j}_e + \omega_{ef}\bar{j}_e j_f\right) = \frac{1}{2}\left(\omega_{0g} - \frac{1}{2}\epsilon_{efg}\omega_{ef}\right)j_g.\end{aligned}\quad (3.10)$$

Explicitly, the  $\omega^\pm$  are given by the formulae

$$\omega^+ = -2\text{Im}\left(e^\vee e(\varphi)\right), \quad \omega^- = -2\text{Im}\left(e(\varphi)e^\vee\right). \quad (3.11)$$

The components of the Riemann 2-form  $R_{ab}$  can be assembled into the two quaternionic 2-forms

$$\begin{aligned}R^+ &= -\frac{1}{4}\left(R_{00} + R_{0f}\bar{j}_f + R_{e0}j_e + R_{ef}j_e\bar{j}_f\right) = \frac{1}{2}\left(R_{0g} + \frac{1}{2}\epsilon_{efg}R_{ef}\right)j_g, \\ R^- &= +\frac{1}{4}\left(R_{00} + R_{0f}j_f + R_{e0}\bar{j}_e + R_{ef}\bar{j}_e j_f\right) = \frac{1}{2}\left(R_{0g} - \frac{1}{2}\epsilon_{efg}R_{ef}\right)j_g.\end{aligned}\quad (3.12)$$

By explicit computation, one finds

$$R^+ = 2\text{Im}\left(e^\vee \wedge (de(\varphi) + 2|e(\varphi)|^2 \bar{e}^\vee)\right), \quad R^- = -2\text{Im}\left((de(\varphi) + 2|e(\varphi)|^2 \bar{e}^\vee) \wedge e^\vee\right). \quad (3.13)$$

The components of the Ricci 1-form  $S_a$  can be organized into the quaternionic 1-form

$$S = S_0 + S_e j_e. \quad (3.14)$$

This is explicitly given by

$$S = -8\left[d\bar{e}(\varphi) + 2(\bar{e}(e(\varphi)) + 3|e(\varphi)|^2)e^\vee\right]. \quad (3.15)$$

Finally, the Ricci scalar  $s$  is given by

$$s = -96\left[\bar{e}(e(\varphi)) + 2|e(\varphi)|^2\right]. \quad (3.16)$$

*Proof.* We give only a sketch. For a conformally flat metric, one has

$$\omega_{ab} = e_b(\varphi)e^\vee_a - e_a(\varphi)e^\vee_b. \quad (3.17)$$

From this relation, using the standard definitions of the Riemann 2-form  $R_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega_{cb}$ , the Ricci 1-form  $S_a = \iota(e_b)R_{ba}$  and the Ricci scalar  $s = \iota(e_a)S_a$ , it is easy to see that

$$R_{ab} = e^\vee_b \wedge Q_a - e^\vee_a \wedge Q_b, \quad (3.18)$$

$$S_a = -2Q_a - Qe^\vee_a, \quad (3.19)$$

$$s = -6Q, \quad (3.20)$$

where

$$Q_a = de_a(\varphi) + \frac{1}{2}e_c(\varphi)e_c(\varphi)e^\vee_a, \quad (3.21)$$

$$Q = e_c(e_c(\varphi)) + 2e_c(\varphi)e_c(\varphi). \quad (3.22)$$

Using these formulae, one obtains straightforwardly the above relations. QED

From (3.11), one can derive the identity

$$de^\vee - \omega^+ \wedge e^\vee + e^\vee \wedge \omega^- = 0, \quad (3.23)$$

which is equivalent to the well-known relation  $de^\vee_a + \omega_{ab} \wedge e^\vee_b = 0$ . From (3.11) and (3.13), one can verify that

$$R^+ = d\omega^+ - \omega^+ \wedge \omega^+, \quad R^- = d\omega^- + \omega^- \wedge \omega^-, \quad (3.24)$$

relations which are equivalent to the definition of the Riemann 2-form  $R_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega_{cb}$ . Other basic relations could be obtained in a similar manner.

Expressions of the Pontryagin density  $\gamma = \frac{1}{8\pi^2}W_{ab} \wedge W_{ab}$ , where  $W_{ab}$  is the Weyl 2-form, and of the Euler density  $\epsilon = \frac{1}{32\pi^2}\epsilon_{abcd}R_{ab} \wedge R_{cd}$  can similarly be obtained. For a locally conformally flat metric such as  $g$ , one obviously has

$$\gamma = 0. \quad (3.25)$$

$\epsilon$  is explicitly given by

$$\begin{aligned} \epsilon = \left(\frac{2}{\pi}\right)^2 & \left\{ 12 \left[ \bar{e}(e(\varphi)) + 2|e(\varphi)|^2 \right]^2 * 1 \right. \\ & \left. - \operatorname{Re} \left[ (d\bar{e}(\varphi) - \bar{e}(e(\varphi))e^\vee) \wedge * (de(\varphi) - \bar{e}(e(\varphi))\bar{e}^\vee) \right] \right\}. \end{aligned} \quad (3.26)$$

*Proof.* It is known that  $\epsilon = \frac{1}{16\pi^2} \left\{ W_{ab} \wedge *W_{ab} + \frac{1}{12}s^2 * 1 - (S_a - \frac{1}{4}se^\vee_a) \wedge * (S_a - \frac{1}{4}se^\vee_a) \right\}$ . In the present case,  $W_{ab} = 0$ , as the metric is locally conformally flat. Using (3.19)–(3.20) and (3.21)–(3.22), it is straightforward to derive the above formula. QED

#### *Global quaternionic Riemannian geometry of a Kulkarni 4-fold*

The quaternionic tensors constructed in the previous subsection have very simple covariance properties on a Kulkarni 4-fold  $X$ .

The matching is implemented by the  $\operatorname{Sp}(1)$  transition functions

$$\theta^\pm_{\alpha\beta} = \eta^\pm_{\alpha\beta} / |\eta^\pm_{\alpha\beta}|, \quad (3.27)$$

with  $\eta^\pm_{\alpha\beta}$  given by (2.23). In general, these do not form a smooth  $\text{Sp}(1)$  1-cocycle, unless  $w = 1$ , as, by (2.31),

$$\theta^\pm_{\alpha\gamma} = w_{\alpha\beta\gamma} \theta^\pm_{\alpha\beta} \theta^\pm_{\beta\gamma}, \quad (3.28)$$

when  $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ . However,  $\theta^\pm_L \otimes \theta^\mp_R$  and  $\theta^\pm_L \otimes \theta^\pm_R$  are, respectively, a  $(\text{Sp}(1) \times \text{Sp}(1))/\mathbb{Z}_2$  1-cocycle and a  $\text{Sp}(1)/\mathbb{Z}_2$  1-cocycle.

We assume that the local scales  $\varphi_\alpha$  match as

$$\varphi_\alpha = \varphi_\beta - \ln |\eta^+_{\alpha\beta}| + \ln |\eta^-_{\alpha\beta}|, \quad (3.29)$$

whenever  $U_\alpha \cap U_\beta \neq \emptyset$ . This is designed in such a way to render  $g = \{g_\alpha\}$  a globally defined metric (see (3.32) below).

The matching relations for the einbein  $e = \{e_\alpha\}$  and  $e^\vee = \{e^\vee_\alpha\}$  are

$$e_\alpha = \theta^-_{\alpha\beta} e_\beta (\theta^+_{\alpha\beta})^{-1} \quad (3.30)$$

and

$$e^\vee_\alpha = \theta^+_{\alpha\beta} e^\vee_\beta (\theta^-_{\alpha\beta})^{-1}, \quad (3.31)$$

with  $U_\alpha \cap U_\beta \neq \emptyset$ .

*Proof.* These relations follow readily from combining (2.24), (2.25) and (3.29) with (3.5) and (3.7). QED

The matching relations of the metric  $g = \{g_\alpha\}$  are by construction

$$g_\alpha = g_\beta. \quad (3.32)$$

on  $U_\alpha \cap U_\beta \neq \emptyset$ . As a consequence, the Hodge star operators  $*_\alpha$  associated with the  $g_\alpha$  match as

$$*_\alpha = *_\beta. \quad (3.33)$$

For  $U_\alpha \cap U_\beta \neq \emptyset$ , the matching relations for the spin connection 1-forms  $\omega^\pm = \{\omega^\pm_\alpha\}$  are

$$\omega^\pm_\alpha = \theta^\pm_{\alpha\beta} \omega^\pm_\beta (\theta^\pm_{\alpha\beta})^{-1} \pm d\theta^\pm_{\alpha\beta} (\theta^\pm_{\alpha\beta})^{-1}. \quad (3.34)$$

The matching relations for the Riemann 2-forms  $R^\pm = \{R^\pm_\alpha\}$ , the Ricci 1-form  $S = \{S_\alpha\}$  and the Ricci scalar  $s = \{s_\alpha\}$  are

$$R^\pm_\alpha = \theta^\pm_{\alpha\beta} R^\pm_\beta (\theta^\pm_{\alpha\beta})^{-1}, \quad (3.35)$$

$$S_\alpha = \theta^+_{\alpha\beta} S_\beta (\theta^-_{\alpha\beta})^{-1}, \quad (3.36)$$

$$s_\alpha = s_\beta. \quad (3.37)$$

So,  $R^\pm \in \Omega^2(X, \theta^\pm_L \otimes \theta^\pm_R)$ ,  $S \in \Omega^1(X, \theta^+_L \otimes \theta^-_R)$  and  $s \in \Omega^0(X)$ .

*Proof.* The matching relation of the dual vierbein  $e^\vee_a = \{e^\vee_{\alpha a}\}$  is of the form

$$e^\vee_{\alpha a} = r_{\alpha\beta ab} e^\vee_{\beta b}, \quad (3.38)$$

where  $r_{\alpha\beta}$  is some smooth  $\text{SO}(4)$  valued function on  $U_\alpha \cap U_\beta$ . Combining (3.6) and (3.38) and comparing with (3.31), one finds

$$r_{\alpha\beta 0a} + r_{\alpha\beta ea} j_e = \theta^+_{\alpha\beta} (\delta_{0a} + \delta_{ea} j_e) (\theta^-_{\alpha\beta})^{-1}. \quad (3.39)$$

As well-known, one has  $\omega_{\alpha ab} = r_{\alpha\beta ac} r_{\alpha\beta bd} \omega_{\beta cd} - dr_{\alpha\beta ac} r_{\alpha\beta bc}$  and  $R_{\alpha ab} = r_{\alpha\beta ac} r_{\alpha\beta bd} R_{\beta cd}$  and  $S_{\alpha a} = r_{\alpha\beta ab} S_{\beta b}$ . Using (3.39) and the definitions (3.10), (3.12) and (3.14), it is straightforward to check that (3.34), (3.35) and (3.36) hold. (3.37) is obvious. *QED*

#### *Selfduality and the Einstein condition*

Selfdual Einstein 4-folds form a broad class of Riemannian 4-folds, which has been intensively studied [14]. Consider a Kulkarni 4-fold  $X$  equipped with the metric  $g$  of the local form (3.8).  $g$  is locally conformally flat and thus trivially selfdual. The Einstein condition, conversely, is non trivial.

The metric  $g$  is Einstein if and only if, locally,

$$d\bar{e}(\varphi) - \bar{e}(e(\varphi))e^\vee = 0. \quad (3.40)$$

The local solution of this equation is

$$e^{-\varphi} = w + 2\text{Re}(\bar{v}q) + u|q|^2, \quad \text{with } u, w \in \mathbb{R}, v \in \mathbb{H}. \quad (3.41)$$

*Proof.* The Einstein condition states that  $S_a - (s/4)e^\vee_a = 0$ . Using the definitions (3.6) and (3.14) and the formulae (3.15) and (3.16), one gets readily (3.40). Explicitly, using (3.5) and (3.7), (3.40) can be cast as

$$d(\partial_{\bar{q}} e^{-\varphi}) - dq \partial_{qL} (\partial_{\bar{q}} e^{-\varphi}) = 0. \quad (3.42)$$

Now, for any smooth  $\mathbb{H}$ -valued function  $f$ , the condition  $df - dq \partial_{qL} f = 0$  restricts  $f$  to be of the form  $f(q) = a + qb$  with  $a, b \in \mathbb{H}$  [7]. Hence, (3.42) entails that

$$\partial_{\bar{q}} e^{-\varphi} = (v + qu)/2, \quad \text{with } u, v \in \mathbb{H}. \quad (3.43)$$

From (3.43), using that  $e^{-\varphi}$  is real valued, one gets

$$de^{-\varphi} = dq(\bar{v} + \bar{u}\bar{q}) + (v + qu)d\bar{q}. \quad (3.44)$$

The integrability condition  $d^2 e^{-\varphi} = 0$  yields the equation  $dq \wedge (u - \bar{u})d\bar{q} = 0$ , which, as is easy to see, entails that  $u - \bar{u} = 0$  or  $u \in \mathbb{R}$ . So,

$$de^{-\varphi} = d\left[2\operatorname{Re}(\bar{v}q) + u|q|^2\right], \quad (3.45)$$

which, upon integration, yields (3.41). *QED*

For  $U_\alpha \cap U_\beta \neq \emptyset$ , we set

$$K_{\alpha\beta} = (|\eta^+_{\alpha\beta}| |\eta^-_{\alpha\beta}|)^{-\frac{1}{2}} T_{\alpha\beta}, \quad (3.46)$$

with  $T_{\alpha\beta}$  defined in (2.22) and  $\eta^\pm_{\alpha\beta}$  given by (2.23). Then,  $K_{\alpha\beta}$  does not depend on the choice of representative of  $T_{\alpha\beta} \in \operatorname{PGL}(2, \mathbb{H})$  in  $\operatorname{GL}(2, \mathbb{H})$ . Further,  $K = \{K_{\alpha\beta}\}$  is a flat  $\operatorname{GL}(2, \mathbb{H})$  1-cochain satisfying relation (2.30) with  $T_{\alpha\beta}$  substituted by  $K_{\alpha\beta}$ . For an Einstein metric of the form (3.41), set

$$M = \begin{pmatrix} u & v \\ \bar{v} & w \end{pmatrix}. \quad (3.47)$$

Then, one has the matching relation

$$M_\beta = K_{\alpha\beta}^\dagger M_\alpha K_{\alpha\beta}. \quad (3.48)$$

*Proof.* In the proof of relation (2.30), it was shown that  $|\eta^+_{\alpha\beta}| |\eta^-_{\alpha\beta}|$  is a positive constant. Using this fact (2.30) and (2.31), it is immediate to see that  $K = \{K_{\alpha\beta}\}$  is a flat  $\operatorname{GL}(2, \mathbb{H})$  1-cochain satisfying (2.30). Independence from choices of representative is evident from the definition (3.46) and from (2.23). The above matching relation follows from (3.29), upon writing

$$e^{-\varphi} = (\bar{q}, 1) \begin{pmatrix} u & v \\ \bar{v} & w \end{pmatrix} \begin{pmatrix} q \\ 1 \end{pmatrix} \quad (3.49)$$

and using (2.22), (1.11) and (2.23). *QED*

This result is interesting. It reduces the problem of finding a locally conformally flat Einstein metric to the problem of finding a flat positive definite section  $M = \{M_\alpha\}$  of the flat 1-cocycle  $\operatorname{Sq}K$ , where, for any  $A \in \operatorname{GL}(2, \mathbb{H})$ ,  $\operatorname{Sq}AU = A^\dagger UA$ , for  $U$  a 2 by 2 matrix on  $\mathbb{H}$ .

*The conformal d'Alembertian  $W$  and the d'Alembertian*  $\square$

Let  $X$  be a Kulkarni 4-fold with the metric  $g$  of eq. (3.8). The conformal d'Alembertian  $W$  of  $g$  is defined by

$$Wf = d * df - \frac{s}{6} f * 1, \quad (3.50)$$



for  $f \in \Omega^0(X)$ . So,  $Wf \in \Omega^4(X)$ .  $W$  is simply related to the operator  $\square$  is defined in (2.39). Indeed,  $e^\varphi f \in \Omega^0(X, \rho)$  and

$$Wf = 16e^\varphi \square(e^\varphi f). \quad (3.51)$$

*Proof.* Combining (2.38) and (3.29), one verifies easily that  $e^\varphi f \in \Omega^0(X, \rho)$  if  $f \in \Omega^0(X)$ . As is well-known, the operator  $W$  is conformally covariant. If  $g_0$  and  $g = e^h g_0$  are two conformally related metrics, then  $Wf = e^h W_0(e^h f)$ . If we take  $g_0$  to be the flat metric and  $g$  to be the metric (3.3), we get (3.51) readily. *QED*

(3.51) entails immediately an isomorphism  $\ker W \cong \ker \square$  of  $\mathbb{R}$  linear spaces.

A well-known argument à la Bochner shows that, if  $X$  is compact and  $s \geq 0$  and  $s \neq 0$  on  $X$ , then  $\dim \ker W = 0$ . So, on a compact Kulkarni 4-fold  $X$  such that the associated conformal class of locally conformally flat metrics contains a representative whose  $s$  has the above properties,  $\dim \ker \square = 0$ , that is there are no harmonic  $F \in \Omega^0(X, \rho)$ .

*The Dirac operator  $\not{D}$  and the Fueter operators  $\bar{\partial}_{R,L}$*

Let  $X$  be a Kulkarni 4-fold with  $w = 1$  equipped with the metric  $g$  of eq. (3.8). We set  $\sigma^+ = \theta^+_R$  and  $\sigma^- = \theta^-_L$ . Owing to (3.28), as  $w = 1$ , the  $\sigma^\pm$  are smooth  $\text{Sp}(1)$  1-cocycles depending on a choice of a flat  $\mathbb{Z}_2$  1-cocycle  $a$ . We set  $\sigma = \sigma^+ \oplus \sigma^-$ . So, any  $\lambda \in \Omega^0(X, \sigma)$  is of the form  $\lambda = \lambda^+ \oplus \lambda^-$  with  $\lambda^\pm \in \Omega^0(X, \sigma^\pm)$ . We set

$$(\lambda_1, \lambda_2) = \text{Re}(\overline{\lambda_1^+} \lambda_2^+) + \text{Re}(\overline{\lambda_1^-} \lambda_2^-), \quad (3.52)$$

for  $\lambda_1, \lambda_2 \in \Omega^0(X, \sigma)$  and, for a vector field  $u$  on  $X$ ,

$$\not{u}\lambda = (\overline{\lambda^-} \langle \bar{e}^\vee, u \rangle) \oplus (\langle \bar{e}^\vee, u \rangle \overline{\lambda^+}), \quad (3.53)$$

for  $\lambda \in \Omega^0(X, \sigma)$ . Then,  $\Omega^0(X, \sigma)$  is a real Clifford module on  $(X, g)$  with Clifford inner product and Clifford action given respectively by (3.52) and (3.53).

*Proof.* If  $\lambda^\pm \in \Omega^0(X, \sigma^\pm)$ , one has  $\lambda^+_\alpha = \lambda^+_\beta \theta^+_{\beta\alpha}$  and  $\lambda^-_\alpha = \theta^-_{\alpha\beta} \lambda^-_\beta$ , whenever defined. Further,  $|\theta^\pm_{\alpha\beta}| = 1$ , by (3.27). Taking these relations into account, one verifies that  $(\lambda_1, \lambda_2)_\alpha = (\lambda_1, \lambda_2)_\beta$ . So, the Clifford inner product is well-defined. Using the same relations once more and (3.31), one verifies also that  $(\not{u}\lambda)^+_\alpha = (\not{u}\lambda)^+_\beta \theta^+_{\beta\alpha}$  and  $(\not{u}\lambda)^-_\alpha = \theta^-_{\alpha\beta} (\not{u}\lambda)^-_\beta$ . So,  $\not{u}$  maps linearly  $\Omega^0(X, \sigma^\pm)$  into  $\Omega^0(X, \sigma^\mp)$ . Finally, one checks easily that  $(\lambda_1, \not{u}\lambda_2) = (\not{u}\lambda_1, \lambda_2)$  and, by using (3.8), that  $\not{u}^2 = g(u, u)1$ . *QED*

For  $\lambda \in \Omega^0(X, \sigma)$ , we define

$$D\lambda = (d\lambda^+ + \lambda^+ \omega^+) \oplus (d\lambda^- + \omega^- \lambda^-). \quad (3.54)$$

Then,  $D$  is a Clifford connection for the Clifford module  $\Omega^0(X, \sigma)$ .

*Proof.* Using (3.34) and the matching relations of  $\lambda^\pm$  given above, it is straightforward to check that  $(D\lambda)^+_\alpha = (D\lambda)^+_\beta \theta^+_{\beta\alpha}$  and  $(D\lambda)^-_\alpha = \theta^-_{\alpha\beta} (D\lambda)^-_\beta$ , whenever defined. So,  $D$  maps  $\Omega^0(X, \sigma^\pm)$  into  $\Omega^1(X, \sigma^\pm)$ .  $D$  manifestly has the properties defining a connection on  $\Omega^0(X, \sigma)$ . From the identity  $\nabla_v e^\vee_a + \langle \omega_{ab}, v \rangle e^\vee_b = 0$ , where  $\nabla$  is the Levi-Civita connection and  $v$  a vector field on  $X$ , and from (3.6) and (3.10), it is straightforward to show that  $\nabla_v e^\vee - \langle \omega^+, u \rangle e^\vee - e^\vee \langle \omega^-, u \rangle = 0$ . Using this latter identity, one checks by simply applying the definitions (3.53) and (3.54) that  $[D, \not{u}] = \nabla \not{u}$ . This shows that  $D$  is a Clifford connection. QED

The Dirac operator  $\not{D}$  associated with the Clifford connection  $D$  of the Clifford module  $\Omega^0(X, \sigma)$  is readily obtained:

$$\not{D}\lambda = \left(4\langle \overline{(D\lambda)}^-, e \rangle\right) \oplus \left(4\langle e, \overline{(D\lambda)}^+ \rangle\right), \quad (3.55)$$

with  $\lambda \in \Omega^0(X, \sigma)$ . This is very simply related to the Fueter operators  $\bar{\partial}_{R,L}$  defined in (2.44). Indeed,  $e^{\frac{3}{2}\varphi} \lambda^\pm \in \Omega^0(X, \varpi^\pm)$  and

$$\not{D}\lambda = \left(4e^{-\frac{5}{2}\varphi} \partial_{\bar{q}L} (e^{\frac{3}{2}\varphi} \lambda^-)\right) \oplus \left(4(\lambda^+ e^{\frac{3}{2}\varphi}) \partial_{\bar{q}R} e^{-\frac{5}{2}\varphi}\right). \quad (3.56)$$

*Proof.* Combining (2.43), (3.27), (3.29) and the matching relations of the  $\lambda^\pm$ , is easily seen that  $e^{\frac{3}{2}\varphi} \lambda^\pm \in \Omega^0(X, \varpi^\pm)$ . From (3.1), (3.2) and (3.17), one has that  $\omega_{ab} = \delta_{ai} \delta_b^j \partial_{xj} \varphi dx^i - \delta_{bj} \delta_a^i \partial_{xi} \varphi dx^j$ . Using this relation, (3.5) and (2.2), it is easy to verify that  $\langle (D\lambda)^+, \bar{e} \rangle = (\lambda^+ \partial_{\bar{q}R} + \frac{3}{2} \lambda^+ \varphi \partial_{\bar{q}L}) e^{-\varphi}$  and  $\langle \bar{e}, (D\lambda)^- \rangle = e^{-\varphi} (\partial_{\bar{q}L} \lambda^- + \frac{3}{2} \partial_{\bar{q}L} \varphi \lambda^-)$ . Using these expressions in (3.55), one gets (3.56) immediately. QED

It follows immediately from (3.56) that  $\ker \bar{\partial}_R \cong \ker \not{D}|_{\Omega^0(X, \sigma^+)}$  and  $\ker \bar{\partial}_L \cong \ker \not{D}|_{\Omega^0(X, \sigma^-)}$ , where the first (second) isomorphism is left (right)  $\mathbb{H}$ -linear.

The Dirac operator  $\not{D}$  satisfies the well-known Bochner–Lichnerowicz–Weitzenboeck formula  $\not{D}^2 = -\square_D + \frac{1}{4}s$ , with  $\square_D$  the d'Alembertian of the Clifford connection  $D$ . By a well-known argument à la Bochner, we see that, if  $X$  is compact and  $s \geq 0$  and  $s \not\equiv 0$  on  $X$ , then  $\dim \ker \not{D} = 0$ . So, on a compact Kulkarni 4-fold  $X$  such that the associated conformal class of locally conformally flat metrics contains a representative whose  $s$  has the above properties,  $\dim \ker \bar{\partial}_{R,L} = 0$ , that is there are no Fueter holomorphic  $\Phi \in \Omega^0(X, \varpi^+)$  and  $\Psi \in \Omega^0(X, \varpi^-)$ .

When  $X$  is compact, one can compute the index of  $\not{D}$ ,  $\text{ind} \not{D}$ , by using the Atiyah–Singer index theorem. One has

$$\text{ind} \not{D} = \dim \ker \not{D}|_{\Omega^0(X, \sigma^+)} - \dim \ker \not{D}|_{\Omega^0(X, \sigma^-)} = 0 \quad (3.57)$$

*Proof.* Using (3.53) and (3.54) and taking (3.24) into account, one finds that  $D^2\lambda - \frac{1}{4}R_{ab}\phi_a\phi_b\lambda = (\lambda^+(d\omega^+ - \omega^+ \wedge \omega^+ - R^+)) \oplus ((d\omega^- + \omega^- \wedge \omega^- - R^-)\lambda^-) = 0$ . The Clifford connection  $D$  has thus no twisting. In this case, the Atiyah–Singer index theorem gives  $\text{ind}\mathcal{D} = -\frac{1}{24}\int_X p_1(X)$ . On account of (2.54),  $\text{ind}\mathcal{D} = 0$ . *QED*

When  $X$  is compact, we conclude from (3.57) that

$$\dim \ker \bar{\partial}_R = \dim \ker \bar{\partial}_L. \quad (3.58)$$

The number of right Fueter holomorphic sections  $\Phi \in \Omega^0(X, \varpi^+)$  equals the number of left Fueter holomorphic sections  $\Psi \in \Omega^0(X, \varpi^-)$ .

*The isometry group of the metric  $g$*

Given a metric  $g$  on  $X$  of the form (3.8), we denote by  $\text{UAut}(X, g)$  the subgroup of  $\text{Aut}(X)$  leaving  $g$  invariant.

*Examples of special metrics*

Below, we shall consider the Kulkarni 4–folds of Kleinian type  $\Gamma \backslash D$ , which were described at the end of section 2.

i)  $D = \mathbb{HP}^1$ .  $\mathbb{HP}^1$  has the distinguished metric

$$g = \frac{4\text{Re}(d\bar{q} \otimes dq)}{(1 + |q|^2)^2}. \quad (3.59)$$

$g$  is nothing but the customary round metric of  $S^4$ . As is well-known,  $g$  is Einstein with  $s = 12$ .  $\text{UAut}(\mathbb{HP}^1, g)$  is the subgroup of  $\text{PGL}(2, \mathbb{H})$  formed by those  $T$  such that  $|T_{11}|^2 + |T_{21}|^2 = |T_{22}|^2 + |T_{12}|^2 = k$  for some  $k \in \mathbb{R}_+$  and  $\bar{T}_{11}T_{12} + \bar{T}_{21}T_{22} = 0$  and is thus a proper subgroup of  $\text{Aut}(\mathbb{HP}^1)$ .

ii)  $D = \mathbb{H}^1$ .  $\mathbb{H}^1$  has the distinguished metric

$$g = 4\text{Re}(d\bar{q} \otimes dq) \quad (3.60)$$

So,  $g$  is the flat euclidean metric of  $\mathbb{R}^4$ .  $\text{UAut}(\mathbb{H}^1, g)$  is the subgroup of  $\text{PGL}(2, \mathbb{H})$  formed by those  $T$  such that  $|T_{11}| = |T_{22}| = 1$  and  $T_{21} = 0$  and is thus a proper subgroup of  $\text{Aut}(\mathbb{H}^1)$ . This metric induces a special metric on each Kulkarni 4–folds  $\Gamma \backslash \mathbb{H}^1$  since, as it is easy to show, every Kleinian group  $\Gamma$  for  $\mathbb{H}^1$  is contained in  $\text{UAut}(\mathbb{H}^1, g)$ .

iii)  $D = B_1(\mathbb{H}^1)$ .  $B_1(\mathbb{H}^1)$  has the distinguished metric

$$g = \frac{4\text{Re}(d\bar{q} \otimes dq)}{(1 - |q|^2)^2}. \quad (3.61)$$

As appears,  $g$  is nothing but the Poincaré metric of  $B_1(\mathbb{R}^4)$ .  $g$  is Einstein with  $s = -12$ . One checks that  $\text{UAut}(B_1(\mathbb{H}^1), g)$  is the subgroup of  $\text{PGL}(2, \mathbb{H})$  formed by those  $T$  such

that  $|T_{11}|^2 - |T_{21}|^2 = |T_{22}|^2 - |T_{12}|^2 = k$  for some  $k \in \mathbb{R}_+$  and  $\bar{T}_{11}T_{12} - \bar{T}_{21}T_{22} = 0$ , so that  $\text{UAut}(B_1(\mathbb{H}^1), g) = \text{Aut}(B_1(\mathbb{H}^1))$ . Therefore, this metric induces a special metric on each Kulkarni 4-folds  $\Gamma \backslash B_1(\mathbb{H}^1)$  for every Kleinian group  $\Gamma$  for  $B_1(\mathbb{H}^1)$ .

*iv)*  $D = \mathbb{H}^1 - \{0\}$ .  $\mathbb{H}^1 - \{0\}$  has the special metric

$$g = \frac{\text{Re}(d\bar{q} \otimes dq)}{|q|^2}. \quad (3.62)$$

One can show that  $\text{UAut}(\mathbb{H}^1 - \{0\}, g) = \text{Aut}(\mathbb{H}^1 - \{0\})$  [9]. Therefore, this metric induces a special metric on each Kulkarni 4-folds  $\Gamma \backslash (\mathbb{H}^1 - \{0\})$  for every Kleinian group  $\Gamma$  for  $\mathbb{H}^1 - \{0\}$ .

*v)*  $D = \mathbb{H}^1 - \mathbb{R}^1$ .  $\mathbb{H}^1 - \mathbb{R}^1$  has the special metric

$$g = \frac{\text{Re}(d\bar{q} \otimes dq)}{|\text{Im} q|^2}. \quad (3.63)$$

It is possible to show that  $\text{UAut}(\mathbb{H}^1 - \mathbb{R}^1) = \text{Aut}(\mathbb{H}^1 - \mathbb{R}^1)$  [9]. Therefore, this metric induces a special metric on each Kulkarni 4-folds  $\Gamma \backslash (\mathbb{H}^1 - \mathbb{R}^1)$  for every Kleinian group  $\Gamma$  for  $\mathbb{H}^1 - \mathbb{R}^1$ .

#### 4. Classical 4-dimensional conformal field theory and Kulkarni geometry

In this section, we consider first some general properties of a classical conformal field theory on a Kulkarni 4-fold  $X$ . Later, we illustrate two basic models, the complex scalar and the Dirac fermion (see [15] for background).

Below, we shall assume that  $X$  is compact. In this way, integrals are convergent and, as  $X$  has no boundary (see sect. 2), integration by parts can be carried out without picking boundary contributions.

##### *The classical action*

The classical action of a conformal field theory on a 4-fold  $X$  is some local functional  $\mathcal{I}(\Phi, e^\vee)$  of a set of conformal fields  $\Phi$  and a dual vierbein  $e^\vee_a$ . By conformal invariance, for any smooth function  $f$  on  $X$ , one has

$$\mathcal{I}(e^{-f\Lambda}\Phi, e^f e^\vee) = \mathcal{I}(\Phi, e^\vee), \quad (4.1)$$

where  $\Lambda$  is the matrix of the conformal weights of the fields  $\Phi$ .

Consider now a conformally flat background  $e^\vee_a$  of the form (3.2). Because of conformal invariance, one has that

$$\mathcal{I}(\Phi, e^\vee) = I(\phi), \quad (4.2)$$

where

$$\phi = e^{\varphi\Lambda}\Phi \quad (4.3)$$

is a conformally invariant field. The functional  $I(\phi)$  depends only on  $\phi$  and the underlying conformal structure.

As  $\varphi$  is defined only locally and the local representations match as in (3.29), the matching relations of the local representations of  $\phi$  are different from those of the local representations of  $\Phi$ . On a Kulkarni 4-fold  $X$ ,  $\phi$  is a section of some vector bundle constructed from the  $\eta^\pm$  such as  $\rho$  and  $\varpi^\pm$ .

*The energy-momentum tensor*

In a classical field theory on a 4-fold  $X$ , the energy-momentum tensor is the 1-form  $\mathcal{T}_a(\Phi, e^\vee)$ ,  $a = 0, 1, 2, 3$ , valued in the orthonormal frame bundle, defined by the variational identity  $\delta_{e^\vee}\mathcal{I} = -\frac{1}{2\pi^2} \int_X \langle \mathcal{T}_a, \delta e_a \rangle * 1$ , where  $\delta_{e^\vee}\Phi = -\frac{1}{4}\Lambda\delta \ln e\Phi$  with  $e = \det e^\vee$  [15]. If the field theory is conformal, the energy-momentum tensor is traceless and thus satisfies

$$\iota(e_a)\mathcal{T}_a = 0. \quad (4.4)$$

The invariance of the classical action  $\mathcal{I}$  under the action of the group of the automorphisms of the orthonormal frame bundle implies that, for classical field configurations solving the classical field equations, the energy-momentum tensor is symmetric and conserved [15]. The symmetry is encoded in the relation

$$\mathcal{T}_a \wedge e^\vee_a = 0. \quad (4.5)$$

The conservation equation can be cast as

$$d * \mathcal{T}_a + \omega_{ab} \wedge * \mathcal{T}_b = 0. \quad (4.6)$$

For a classical conformal field theory, one has

$$\mathcal{T}_a(e^{-f\Lambda}\Phi, e^f e^\vee) = e^{-3f} \mathcal{T}_a(\Phi, e^\vee), \quad (4.7)$$

for any smooth function  $f$ . This is an immediate consequence of the conformal invariance of the action (eq. (4.1)) and of the definition of  $\mathcal{T}_a$ . Consequently, in a locally conformally flat metric background  $e^\vee_a$  of the form (3.2), one has that

$$\mathcal{T}_a(\Phi, e^\vee) = \delta_{ai} e^{-3\varphi} T_i(\phi), \quad (4.8)$$

where the  $T_i(\phi)$ ,  $i = 0, 1, 2, 3$  are 1-forms depending only on  $\phi$  and the underlying conformal structure. They can be assembled into the quaternionic field

$$T = \frac{1}{4}(T_0 - T_r j_r). \quad (4.9)$$

Then, it is simple to verify that the tracelessness relation (4.4) takes the form

$$\text{Re}(T\iota(\partial_{\bar{q}})) = 0. \quad (4.10)$$

For classical field configurations, the symmetry relation (4.5) reads as

$$\text{Re}(dq \wedge T) = 0, \quad (4.11)$$

while, more importantly, the conservation equation (4.6) becomes simply

$$d \star T = 0. \quad (4.12)$$

This equation no longer contains any explicit dependence on the scale  $\varphi$  of the metric background. Its validity depends crucially on the tracelessness and symmetry relations (4.10) and (4.11).

*Proof.* (4.10) and (4.11) are trivial consequences of (4.4) and (4.5) following from (4.9), (3.1), (3.2), (3.5) and (3.7). (4.12) follows from substituting (3.9), (3.17) and (4.8) into (4.6) upon using (3.1)–(3.2) and (4.4)–(4.5). *QED*

On a Kulkarni 4-fold  $X$ ,  $T \in \Omega^1(X, \zeta_3)$ , where  $\zeta_3$  is given by (2.34).

*Proof.* By (4.8) and (4.9), one has

$$T_\alpha = \frac{e^{3\varphi_\alpha}}{4}(\mathcal{T}_{\alpha 0} - \mathcal{T}_{\alpha e} j_e). \quad (4.13)$$

Now, on  $U_\alpha \cap U_\beta \neq \emptyset$ , one has

$$\mathcal{T}_{\alpha a} = r_{\alpha\beta ab} \mathcal{T}_{\beta b}, \quad (4.14)$$

where  $r_{\alpha\beta}$  is the same  $\text{SO}(4)$  valued function as that appearing in (3.38). Combining (3.27), (3.29), (3.39) and (4.14) and recalling (2.34), one checks easily that the matching relation of the  $T_\alpha$  is the required one. *QED*

In general, for an object of the same tensor type as  $T$ , the conservation equation (4.12) would not be covariant. In the present case, it is thanks to the tracelessness and symmetry properties (4.10)–(4.11).

*The U(1) current*

In a classical field theory with a  $U(1)$  symmetry, the  $U(1)$  current is the 1-form  $\mathcal{J}(\Phi, e^\vee)$  defined by the variational condition  $\delta_\Phi \mathcal{I}|_{\delta\Phi=if\Phi} = -\frac{1}{2\pi^2} \int_X \mathcal{J} \wedge *df$  for any function  $f$  [15]. For classical field configurations solving the classical field equations,  $\mathcal{J}$  satisfies the conservation equation

$$d * \mathcal{J} = 0. \quad (4.15)$$

For a classical conformal field theory, one has

$$\mathcal{J}(e^{-f\Lambda}\Phi, e^f e^\vee) = e^{-2f} \mathcal{J}(\Phi, e^\vee), \quad (4.16)$$

for any smooth function  $f$ . This is an immediate consequence of the conformal invariance of the action (eq. (4.1)) and of the definition of  $\mathcal{J}$ . In the locally conformally flat metric background  $e^\vee_a$  of eq. (3.2), one has then

$$\mathcal{J}(\Phi, e^\vee) = e^{-2\varphi} J(\phi), \quad (4.17)$$

where  $J(\phi)$  is a 1-form depending only on  $\phi$  and the underlying conformal structure. The conservation equation (4.15) takes then the form

$$d \star J = 0. \quad (4.18)$$

*Proof.* This follows readily from (4.15) upon combining (3.9) and (4.17). *QED*

This equation no longer contains any explicit dependence on the scale  $\varphi$  of the metric background.

If  $X$  is a Kulkarni 4-fold,  $J \in \Omega^1(X, \rho^2)$ , where  $\rho$  is defined in (2.38).

*Proof.* Immediate from (3.29) and (4.17). *QED*

Then, by (2.29),  $\star J \in \Omega^3(X)$ . The conservation equation (4.18) is thus manifestly covariant.

### *The biquaternion algebra*

The models examined below involve the complexification of the quaternion field  $\mathbb{H}$ , the complex biquaternion algebra  $\mathbb{H} \otimes \mathbb{C}$ . In this brief algebraic interlude, we recall a few basic facts about  $\mathbb{H} \otimes \mathbb{C}$  and introduce basic notation.

Here and below, to avoid possible confusion with the corresponding quaternionic operations, we denote complex conjugation by  $\bar{\phantom{x}}_c$  and complex real (imaginary) part by  $\text{Re}_c$  ( $\text{Im}_c$ ).

A generic element  $z \in \mathbb{H} \otimes \mathbb{C}$  can be represented as a real linear combination of elements of the form  $a \otimes \zeta$ , where  $a \in \mathbb{H}$  and  $\zeta \in \mathbb{C}$ . As a complex algebra,  $\mathbb{H} \otimes \mathbb{C}$

carries a conjugation  $\bar{\cdot}$  defined by  $\overline{a \otimes \zeta} = \bar{a} \otimes \bar{\zeta}$  and an antilinear involution  $\sim$  defined by  $\widetilde{a \otimes \zeta} = a \otimes \bar{\zeta}$ <sup>5</sup>.  $\mathbb{H}$  can be canonically identified with the subalgebra of  $\mathbb{H} \otimes \mathbb{C}$  fixed by  $\sim$ . The action of the conjugation  $\bar{\cdot}$  on this subalgebra coincides with the quaternionic conjugation  $\bar{\cdot}$  as defined earlier.

There is a canonical algebra isomorphism  $c : \mathbb{C}(2) \rightarrow \mathbb{H} \otimes \mathbb{C}$ , where  $\mathbb{C}(2)$  is the complex algebra of 2 by 2 complex matrices. Denoting by  $\tau_f$ ,  $f = 1, 2, 3$ ,  $-i$  times the standard Pauli matrices,  $c$  is uniquely defined by  $c(1_2) = 1 \otimes 1$  and  $c(\tau_f) = j_f \otimes 1$ . The isomorphism  $c$  has the properties that  $\det M = \tilde{c}(M)c(M)$  and that  $c(M_c^\dagger) = \bar{c}(M)$  and  $c(C^{-1}\bar{M}_c C) = \tilde{c}(M)$  for any  $M \in \mathbb{C}(2)$ , where  $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  is the conjugation matrix.

*The complex scalar*

Consider a complex scalar field  $\Phi$  with action

$$\mathcal{I}(\Phi, \bar{\Phi}_c, e^\vee) = \frac{1}{2\pi^2} \int_X d^4x g^{\frac{1}{2}} \left[ g^{ij} \partial_i \bar{\Phi}_c \partial_j \Phi + \frac{1}{6} s \bar{\Phi}_c \Phi \right], \quad (4.19)$$

where  $g$  is the metric corresponding to  $e^\vee_a$  and  $s$  is the Ricci scalar. The field  $\Phi$  has conformal weight  $\Lambda = 1$ . It is well-known that the above action is conformally invariant [15].

The conformally invariant field  $\phi$  corresponding to  $\Phi$  is thus given by

$$\phi = e^\varphi \Phi. \quad (4.20)$$

Then,  $\phi \in \Omega^0(X, \rho) \otimes \mathbb{C}$ , where  $\rho$  is defined in (2.38).

*Proof.* By (3.29).

*QED*

In terms of  $\phi$ , the action functional is simply

$$I(\phi, \bar{\phi}_c) = -\frac{8}{\pi^2} \int_X \bar{\phi}_c \square \phi, \quad (4.21)$$

where  $\square$  is defined in (2.39). The integrand belongs to  $\Omega^4(X)$ , as  $\square \phi \in \Omega^0(X, \rho^{-1}) \otimes \mathbb{C}$ , and integration is thus well-defined.

*Proof.* This follows from substituting (3.20) and (3.22), upon using (3.1), and (4.20) into (4.19), by a straightforward calculation.

*QED*

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<sup>5</sup> A conjugation (antilinear involution)  $K$  on a complex algebra  $A$  is an antilinear map  $K : A \rightarrow A$  such that  $K^2 = 1_A$  and that, for  $a, b \in A$ ,  $K(ab) = K(b)K(a)$  ( $K(ab) = K(a)K(b)$ ).



The classical field equations of  $\Phi$  are [15]

$$\nabla^j \nabla_j \Phi - \frac{1}{6} s \Phi = 0. \quad (4.22)$$

In terms of the field  $\phi$ , they read simply as

$$\square \phi = 0, \quad (4.23)$$

that is  $\phi$  is harmonic. See the discussion of section 3 concerning the solutions of this equation.

The energy–momentum tensor of the complex scalar  $\Phi$  is given by [15]

$$\begin{aligned} \mathcal{T}_a(\Phi, \bar{\Phi}_c, e^\vee) = \text{Re}_c \left\{ \frac{2}{3} \left[ \bar{\Phi}_c e_a^j \nabla_j \nabla_i \Phi - \frac{1}{4} \bar{\Phi}_c \nabla^k \nabla_k \Phi e^\vee_{ai} \right] \right. \\ \left. - \frac{4}{3} \left[ e_a^j \nabla_j \bar{\Phi}_c \nabla_i \Phi - \frac{1}{4} \nabla^k \bar{\Phi}_c \nabla_k \Phi e^\vee_{ai} \right] - \frac{1}{3} \left[ S_{ai} - \frac{1}{4} s e^\vee_{ai} \right] \bar{\Phi}_c \Phi \right\} dx^i. \end{aligned} \quad (4.24)$$

One can verify that (4.8) holds. The conformally invariant energy–momentum tensor  $T$  is given by

$$\begin{aligned} T(\phi, \bar{\phi}_c) = -\frac{2}{3} \left\{ \partial_q \bar{\phi}_c d\phi - \frac{1}{2} \bar{\phi}_c d\partial_q \phi + \partial_q \phi d\bar{\phi}_c - \frac{1}{2} \phi d\partial_q \bar{\phi}_c \right. \\ \left. - \left( \partial_{\bar{q}} \bar{\phi}_c \partial_q \phi - \frac{1}{2} \bar{\phi}_c \partial_{\bar{q}} \partial_q \phi + \partial_{\bar{q}} \phi \partial_q \bar{\phi}_c - \frac{1}{2} \phi \partial_{\bar{q}} \partial_q \bar{\phi}_c \right) d\bar{q} \right\}. \end{aligned} \quad (4.25)$$

We have checked that  $T$  satisfies (4.10) and that (4.11) and (4.12) hold, when  $\phi$  fulfills the field equations (4.23). For a field configuration  $\phi$  satisfying (4.23), the second and fourth term proportional to  $d\bar{q}$  in (4.25) are zero.

The model considered has an obvious  $U(1)$  symmetry. The corresponding  $U(1)$  current is

$$\mathcal{J}(\Phi, \bar{\Phi}_c, e^\vee) = 2\text{Im}_c(\Phi \partial_i \bar{\Phi}_c) dx^i. \quad (4.26)$$

It is easy to see that (4.17) is fulfilled with

$$J(\phi, \bar{\phi}_c) = \frac{1}{i} (\phi d\bar{\phi}_c - \bar{\phi}_c d\phi). \quad (4.27)$$

One verifies readily that  $J$  satisfies (4.18), when  $\phi$  satisfies the field equations (4.23).

#### *The Dirac fermion*

Suppose that  $w = 1$ , so that  $X$  is spin, and let us fix the spin structure. Consider a euclidean Dirac fermion field  $\Psi$ .  $\Psi \in \Pi\Omega^0(X, \Sigma^+ \oplus \Sigma^-)$ , where  $\Sigma^\pm$  are the positive/negative chirality spinor bundles and the notation  $\Pi V$  indicates the Grassmann odd partner of a vector space  $V$ .

The Dirac action is

$$\mathcal{I}(\Psi, \Psi_c^\dagger, e^\vee) = \frac{1}{2\pi^2} \int_X d^4x e i \Psi_c^\dagger \gamma_a e_a^j D_j \Psi, \quad (4.28)$$

where  $D$  is the spin covariant derivative,  $D_j \Psi = (\partial_j + \frac{1}{4} \omega_{abj} \gamma_a \gamma_b) \Psi$ , the  $\gamma_a$ ,  $a = 0, 1, 2, 3$ , being the euclidean gamma matrices satisfying  $\gamma_a \gamma_b + \gamma_b \gamma_a = 2\delta_{ab}$  and  $\gamma_a^\dagger = \gamma_a$ . The field  $\Psi$  has conformal weight  $\Lambda = 3/2$ . As is well-known, the above action is conformally invariant [15]. We shall write the action in a way such that its connection with the underlying Kulkarni geometry becomes manifest.

Fix  $v_0 \in \mathbb{C}^2$ ,  $v_0 \neq 0$ . We define a linear map  $Q : \mathbb{C}^2 \rightarrow \mathbb{H} \otimes \mathbb{C}$  by

$$Q(v) = c(|v_0|^{-2} v \otimes v_0^\dagger), \quad v \in \mathbb{C}^2, \quad (4.29)$$

where  $c$  has been defined earlier.

The Dirac fermion field  $\Psi$  can be thought of as a pair of Weyl fermion fields  $(\Psi^+, \Psi^-)$  with  $\Psi^\pm \in \Pi\Omega^0(X, \Sigma^\pm)$ . We set

$$\psi^+ = e^{\frac{3}{2}\varphi} \tilde{Q}(\Psi^+), \quad \psi^- = e^{\frac{3}{2}\varphi} Q(\Psi^-). \quad (4.30)$$

Then,  $\psi^\pm \in \Pi(\Omega^0(X, \varpi^\pm) \otimes \mathbb{C})$ , where the  $\varpi^\pm$  are defined in (2.43).

*Proof.*  $SU(2)$  corresponds precisely via  $c$  to the group  $Sp(1)$  of unit length quaternions in  $\mathbb{H}$ . Further, as  $\det U = 1$  and  $U = C^{-1} \bar{U}_c C = U^{-1} \dagger_c$  for  $U \in SU(2)$ , one has  $c(U) = \tilde{c}(U) = \bar{c}(U)^{-1}$  whenever  $U \in SU(2)$ . Now, comparing the basic relation

$$r_{\alpha\beta 0a} 1_2 + r_{\alpha\beta ea} \tau_e = \Sigma^+_{\alpha\beta} (\delta_{0a} 1_2 + \delta_{ea} \tau_e) (\Sigma^-_{\alpha\beta})^{-1}, \quad (4.31)$$

satisfied by  $\Sigma^\pm_{\alpha\beta}$ , and the relation

$$r_{\alpha\beta 0a} 1_2 + r_{\alpha\beta ea} \tau_e = c^{-1} (\theta^+_{\alpha\beta}) (\delta_{0a} 1_2 + \delta_{ea} \tau_e) c^{-1} ((\theta^-_{\alpha\beta})^{-1}), \quad (4.32)$$

following from (3.39), and recalling that  $c^{-1}(\theta^\pm_{\alpha\beta}) \in SU(2)$  as  $\theta^\pm_{\alpha\beta}$  is  $Sp(1)$  valued, one concludes that

$$c(\Sigma^\pm_{\alpha\beta}) = \theta^\pm_{\alpha\beta}, \quad (4.33)$$

provided the spin structure entering into the definition of  $\theta^\pm$  is suitably chosen. Now, from (4.29), one has that  $Q(Uv) = c(U)Q(v)$  and  $\tilde{Q}(Uv) = \tilde{Q}(v)c(U)^{-1}$  for  $U \in SU(2)$  and  $v \in \Pi\mathbb{C}^2$ . Hence,

$$\begin{aligned} \tilde{Q}(\Psi^+_\alpha) &= \tilde{Q}(\Sigma^+_{\alpha\beta} \Psi^+_\beta) = \tilde{Q}(\Psi^+_\beta) c(\Sigma^+_{\beta\alpha}) = \tilde{Q}(\Psi^+_\beta) \theta^+_{\beta\alpha}, \\ Q(\Psi^-_\alpha) &= Q(\Sigma^-_{\alpha\beta} \Psi^-_\beta) = c(\Sigma^-_{\alpha\beta}) Q(\Psi^-_\beta) = \theta^-_{\alpha\beta} Q(\Psi^-_\beta). \end{aligned} \quad (4.34)$$

From here, it is easy to show the statement combining (4.30) and (3.29) and (3.27). *QED*

In terms of  $\psi^\pm$ , the action functional can be written as <sup>6</sup>

$$\begin{aligned} I(\psi^+, \psi^-, \tilde{\psi}^+, \tilde{\psi}^-) &= |v_0|^2 \frac{2}{\pi^2} \text{Re} \int_X \left[ \tilde{\psi}^+ \bar{\partial}_R \wedge \star dq \psi^- - \psi^+ \bar{\partial}_R \wedge \star dq \tilde{\psi}^- \right] \\ &= |v_0|^2 \frac{2}{\pi^2} \text{Re} \int_X \left[ \tilde{\psi}^+ \star dq \wedge \bar{\partial}_L \psi^- - \psi^+ \star dq \wedge \bar{\partial}_L \tilde{\psi}^- \right]. \end{aligned} \quad (4.35)$$

*Proof.* Using (3.17) and the formulae

$$\gamma_0 = i \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}, \quad \gamma_e = i \begin{pmatrix} 0 & \tau_e \\ -\tau_e^\dagger & 0 \end{pmatrix}, \quad (4.36)$$

one can cast the action integral (4.28) as

$$\begin{aligned} 2\pi^2 \mathcal{I}(\Psi, \Psi_c^\dagger, e^\vee) &= +4\text{Re}_c \int_X \left[ (e^{\frac{3}{2}\varphi} \Psi^{+\dagger}_c) (\partial_0 1_2 + \partial_i \tau_i)_R e^{\frac{3}{2}\varphi} \Psi^- \right] \star 1 \\ &= -4\text{Re}_c \int_X \left[ e^{\frac{3}{2}\varphi} \Psi^{+\dagger}_c (\partial_0 1_2 + \partial_i \tau_i)_L (e^{\frac{3}{2}\varphi} \Psi^-) \right] \star 1. \end{aligned} \quad (4.37)$$

One can show that  $\text{Re}_c(v_2^\dagger v_1) = |v_0|^2 \text{Re} \left( \bar{Q}(v_2) Q(v_1) - \tilde{Q}(v_2) \tilde{Q}(v_1) \right)$  for  $v_1, v_2 \in \Pi\mathbb{C}^2$  and that  $Q(Uv) = c(U)Q(v)$  and  $\tilde{Q}(Uv) = c(U)\tilde{Q}(v)$  for  $U \in \text{SU}(2)$  and  $v \in \Pi\mathbb{C}^2$ . From here, using the relations (2.2), (2.8), (2.9) and (2.44) and the definition (4.30), one gets the above result. *QED*

The classical field equations of  $\Psi$  are [15]

$$\gamma_a e_a^j D_j \Psi = 0. \quad (4.38)$$

In terms of  $\psi^\pm$ , they read simply as

$$\psi^+ \bar{\partial}_R = 0, \quad \bar{\partial}_L \psi^- = 0. \quad (4.39)$$

Hence,  $\psi^+$  ( $\psi^-$ ) is right (left) Fueter holomorphic. See the discussion of section 3 concerning the solutions of these equations.

The energy-momentum tensor of the Dirac fermion  $\Psi$  is [15]

$$\begin{aligned} \mathcal{T}_a(\Psi, \Psi_c^\dagger, e^\vee) &= \text{Re}_c \left\{ \frac{1}{2i} \Psi_c^\dagger \left[ \gamma_b e^\vee_{bj} e_a^k D_k + \gamma_a D_j \right. \right. \\ &\quad \left. \left. - \frac{1}{2} e^\vee_{aj} \gamma_b e_b^k D_k + \frac{1}{2} [\gamma_a, \gamma_c] e^\vee_{cj} \gamma_b e_b^k D_k \right] \Psi \right\} dx^j. \end{aligned} \quad (4.40)$$

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<sup>6</sup> The relative minus sign is due to the anticommuting nature of the fields  $\Psi^\pm$ .

It is straightforward though a bit lengthy to check that (4.8) holds. The conformally invariant energy–momentum tensor  $T$  can be computed. One finds

$$\begin{aligned}
T(\psi^+, \psi^-, \tilde{\psi}^+, \tilde{\psi}^-) &= \frac{|v_0|^2}{4} \left\{ -\tilde{\psi}_- d\psi_+ + \psi_- d\tilde{\psi}_+ + d\tilde{\psi}_- \psi_+ - d\psi_- \tilde{\psi}_+ \right. \\
&\quad + d\bar{q}(\tilde{\psi}_+ \bar{\psi}_- - \bar{\psi}_+ \tilde{\psi}_-) \partial_{qR} + (\psi_- \tilde{\psi}_+ - \tilde{\psi}_- \psi_+) \partial_{\bar{q}R} d\bar{q} \\
&\quad + \frac{3}{2} \left( \partial_{qL} \bar{\psi}_+ \tilde{\psi}_- - \partial_{qL} \tilde{\psi}_+ \bar{\psi}_- + \tilde{\psi}_- \psi_+ \partial_{\bar{q}R} - \psi_- \tilde{\psi}_+ \partial_{\bar{q}R} \right) d\bar{q} \\
&\quad + \frac{1}{2} d\bar{q} \left( \bar{\psi}_+ \tilde{\psi}_- \partial_{qR} - \tilde{\psi}_+ \bar{\psi}_- \partial_{qR} + \partial_{\bar{q}L} \tilde{\psi}_- \psi_+ - \partial_{\bar{q}L} \psi_- \tilde{\psi}_+ \right) \Big\} \\
&= \frac{|v_0|^2}{4} \left\{ -\tilde{\psi}_- d\psi_+ + \psi_- d\tilde{\psi}_+ + d\tilde{\psi}_- \psi_+ - d\psi_- \tilde{\psi}_+ \right. \\
&\quad - \partial_{qL}(\tilde{\psi}_+ \bar{\psi}_- - \bar{\psi}_+ \tilde{\psi}_-) d\bar{q} - d\bar{q} \partial_{\bar{q}L}(\psi_- \tilde{\psi}_+ - \tilde{\psi}_- \psi_+) \\
&\quad - \frac{1}{2} \left( \partial_{qL} \bar{\psi}_+ \tilde{\psi}_- - \partial_{qL} \tilde{\psi}_+ \bar{\psi}_- + \tilde{\psi}_- \psi_+ \partial_{\bar{q}R} - \psi_- \tilde{\psi}_+ \partial_{\bar{q}R} \right) d\bar{q} \\
&\quad \left. - \frac{3}{2} d\bar{q} \left( \bar{\psi}_+ \tilde{\psi}_- \partial_{qR} - \tilde{\psi}_+ \bar{\psi}_- \partial_{qR} + \partial_{\bar{q}L} \tilde{\psi}_- \psi_+ - \partial_{\bar{q}L} \psi_- \tilde{\psi}_+ \right) \right\}.
\end{aligned} \tag{4.41}$$

We have checked that  $T$  fulfills (4.10) and that (4.11) and (4.12) hold, when the  $\psi^\pm$  fulfills the field equations (4.39). For a field configuration  $\psi^\pm$  satisfying (4.39), the terms proportional to  $d\bar{q}$  vanish identically, simplifying the above expressions.

The Dirac action has an obvious U(1) symmetry. The corresponding U(1) current is

$$\mathcal{J}(\Psi, \Psi_c^\dagger, e^\vee) = \Psi_c^\dagger \gamma_a e^\vee_{aj} \Psi dx^j. \tag{4.42}$$

It is easy to see that (4.17) is fulfilled with

$$J(\psi^+, \psi^-, \tilde{\psi}^+, \tilde{\psi}^-) = |v_0|^2 \text{Re} \left( i\tilde{\psi}_+ dq\psi_- + i\psi_+ dq\tilde{\psi}_- \right). \tag{4.43}$$

$J$  satisfies (4.18), when the  $\psi^\pm$  satisfies the field equations (4.39).

## 5. Quantum 4–dimensional conformal field theory and Kulkarni geometry

In this section, we consider first some general properties of a conformal quantum field theory on a Kulkarni 4–fold  $X$  concentrating on the quantum energy–momentum tensor. We then analyze the properties of the operator product expansions for the simple free models studied in the previous section.

Below, we shall assume that  $X$  is compact.

In a quantum 4–dimensional conformal field theory, the local classical action  $\mathcal{I}(\Phi, e^\vee)$  is affected by quantum corrections. The resulting effective action  $\mathcal{I}_e(\Phi, e^\vee)$  is a non local

functional of  $\Phi$  and  $e^\vee_a$ . In general,  $\mathcal{I}_e(\Phi, e^\vee)$  is no longer conformally invariant but, conversely, suffers an additive conformal anomaly. We assume that, for any smooth function  $f$ ,

$$\mathcal{I}_e(e^{-f\Lambda}\Phi, e^f e^\vee) = \mathcal{I}_R(f, e^\vee) + \mathcal{I}_e(\Phi, e^\vee), \quad (5.1)$$

where  $\mathcal{I}_R(f, e^\vee)$  is the Riegert action, which is local and independent from  $\Phi$  [16–17].

*The quantum energy–momentum tensor*

One can define the energy–momentum tensors  $\mathcal{T}_{ea}(\Phi, e^\vee)$  and  $\mathcal{T}_{Ra}(f, e^\vee)$  for the actions  $\mathcal{I}_e$  and  $\mathcal{I}_R$  in the same way as done in the classical case:  $\delta_{e^\vee}\mathcal{I}_e = -\frac{1}{2\pi^2} \int_X \langle \mathcal{T}_{ea}, \delta e_a \rangle * 1$  and  $\delta_{e^\vee}\mathcal{I}_R = -\frac{1}{2\pi^2} \int_X \langle \mathcal{T}_{Ra}, \delta e_a \rangle * 1$ , where  $\delta_{e^\vee} f = -\frac{1}{4}\delta \ln e$ . Because of the conformal anomaly,  $\mathcal{T}_{ea}$  and  $\mathcal{T}_{Ra}$  do not satisfy a condition of tracelessness analogous to (4.4). However, since invariance under the automorphism group of the orthonormal frame bundle is not anomalous,  $\mathcal{T}_{ea}$  still satisfies (4.5) and (4.6) in the vacuum, i. e. at vanishing field configurations. So,  $\mathcal{T}_{ea}|_{\Phi=0}$  is symmetric,

$$\mathcal{T}_{ea} \wedge e^\vee_a|_{\Phi=0} = 0, \quad (5.2)$$

and satisfies the Ward identity

$$(d * \mathcal{T}_{ea} + \omega_{ab} \wedge * \mathcal{T}_{eb})|_{\Phi=0} = 0. \quad (5.3)$$

$\mathcal{T}_{Ra}$  is also symmetric

$$\mathcal{T}_{Ra} \wedge e^\vee_a = 0, \quad (5.4)$$

while its Ward identity reads

$$d * \mathcal{T}_{Ra} + \omega_{ab} \wedge * \mathcal{T}_{Rb} + \left( \frac{1}{4} d\mathcal{C}_R - \mathcal{C}_R df \right) \wedge * e^\vee_a = 0, \quad (5.5)$$

where the functional  $\mathcal{C}_R(f, e^\vee)$  is defined by  $\delta_f \mathcal{I}_R = +\frac{1}{2\pi^2} \int_X \mathcal{C}_R \delta f * 1$ . The origin of the extra terms in the Ward identity (5.5) is easily understood. If  $\mathcal{I}_R$  were the classical action of some automorphism invariant field theory, they would be absent for a field  $f$  satisfying the classical field equation  $\mathcal{C}_R = 0$  and (5.5) would be analogous to (4.6).

There is another piece of information that is relevant and does not follow directly from (5.1). One has

$$\mathcal{C}_R(0, e^\vee) = 0 \quad \text{on any open set of } X \text{ where } R_{ab} = 0. \quad (5.6)$$

This identity can be justified by noting that, on dimensional grounds,  $\mathcal{C}_R(0, e^\vee)$  is the sum of two contributions. The first is quadratic in the components of the Riemann 2–form  $R_{ab}$

and the derived forms. The second is proportional to  $d * ds$ , where  $s$  is the Ricci scalar. Both contributions vanish in the regions where the background  $e^\vee_a$  is flat.

Because of the anomalous breaking of conformal invariance in the quantum theory,  $\mathcal{T}_{ea}$  does not satisfies a relation of the form (4.7) in the locally conformally flat background of eq. (3.2) and therefore it does not have a structure like that exhibited in (4.8). However, it is still possible extract from  $T_{ea}$  a part  $T_{ei}(\phi)$  depending only on  $\phi$  and the conformal geometry of the base manifold  $X$ . Indeed,

$$\mathcal{T}_{ea}(\Phi, e^\vee) = e^{-3\varphi} \left[ \delta_{ai} T_{ei}(\phi) + \mathcal{L}_{Ra}(\varphi, e^{-\varphi} e^\vee) \right], \quad (5.7)$$

where

$$\mathcal{L}_{Ra} = \mathcal{T}_{Ra} + \frac{1}{4} \mathcal{C}_R e^\vee_a. \quad (5.8)$$

Since the action  $\mathcal{I}_R$  is local,  $\mathcal{T}_{Ra}$  and  $\mathcal{C}_R$  are local expression in the fields  $f$  and  $e^\vee_a$  involving no integration on  $X$ . They are therefore defined also when  $f$  and  $e^\vee_a$  are replaced by the local scale  $\varphi$  and the local dual vierbein  $e^{-\varphi} e^\vee_a$ . The covariance of the composite fields obtained in this way is however quite different from the original one, as will be shown in a moment. Now, one can verify that  $T_{ei}(\phi)$  is conformally invariant, as suggested by the notation. Following (4.9), one sets

$$T_e = \frac{1}{4} (T_{e0} - T_{er} j_r). \quad (5.9)$$

Then, one can verify that  $T_e$  is traceless:

$$\text{Re} (T_e \iota(\partial_{\bar{q}})) = 0. \quad (5.10)$$

Further, in the vacuum, i. e. when  $\phi = 0$ ,  $T_e$  is symmetric and conserved, so that

$$\text{Re} (dq \wedge T_e)|_{\phi=0} = 0 \quad (5.11)$$

and

$$d \star T_e|_{\phi=0} = 0. \quad (5.12)$$

*Proof.* We give only a sketch of the proof. By varying (5.1) with respect to  $f$  and  $e^\vee_a$ , one obtains

$$e^{3f} \langle \mathcal{T}_{ea}(e^{-f\Lambda} \Phi, e^f e^\vee), e_a \rangle - \mathcal{C}_R(f, e^\vee) = 0, \quad (5.13)$$

$$e^{3f} \mathcal{T}_{ea}(e^{-f\Lambda} \Phi, e^f e^\vee) - \mathcal{T}_{ea}(\Phi, e^\vee) - \mathcal{T}_{Ra}(f, e^\vee) - \frac{1}{4} \mathcal{C}_R(f, e^\vee) e^\vee_a = 0. \quad (5.14)$$

From (5.1), it follows that the action  $\mathcal{I}_R$  satisfies the so called 1-cocycle relation

$$\mathcal{I}_R(f_1 + f_2, e^\vee) - \mathcal{I}_R(f_1, e^{f_2} e^\vee) - \mathcal{I}_R(f_2, e^\vee) = 0, \quad (5.15)$$

for any two smooth functions  $f_1, f_2$ . By varying this identity with respect to  $f_1, f_2$  and  $e^\vee_a$ , one obtains

$$\mathcal{C}_R(f_1 + f_2, e^\vee) - e^{4f_2} \mathcal{C}_R(f_1, e^{f_2} e^\vee) = 0, \quad (5.16)$$

$$e^{3f_2} \langle \mathcal{T}_{Ra}(f_1, e^{f_2} e^\vee), e_a \rangle + \mathcal{C}_R(f_2, e^\vee) = 0, \quad (5.17)$$

$$\mathcal{T}_{Ra}(f_1 + f_2, e^\vee) - e^{3f_2} \mathcal{T}_{Ra}(f_1, e^{f_2} e^\vee) - \mathcal{T}_{Ra}(f_2, e^\vee) - \frac{1}{4} \mathcal{C}_R(f_2, e^\vee) e^\vee_a = 0. \quad (5.18)$$

Define  $\mathcal{E}_a(\phi, \varphi, e^\vee) = e^{3\varphi} \mathcal{T}_{ea}(e^{-\varphi\Lambda} \phi, e^\vee) - \mathcal{L}_{Ra}(\varphi, e^{-\varphi} e^\vee)$ . Using (5.14) with  $\Phi$  substituted by  $e^{-\varphi\Lambda} \phi$  and (5.16) and (5.18) with  $f_1, f_2$  and  $e^\vee_a$  substituted by  $f, \varphi$  and  $e^{-\varphi} e^\vee_a$ , respectively, one verifies that  $\mathcal{E}_a(\phi, \varphi + f, e^f e^\vee) = \mathcal{E}_a(\phi, \varphi, e^\vee)$ , showing the conformal invariance of  $\mathcal{E}_a(\phi, \varphi, e^\vee)$ . Thus,

$$T_{ei}(\phi) = \delta_{ia} \left[ e^{3\varphi} \mathcal{T}_{ea}(e^{-\varphi\Lambda} \phi, e^\vee) - \mathcal{L}_{Ra}(\varphi, e^{-\varphi} e^\vee) \right] \quad (5.19)$$

depends only on  $\phi$  and the background conformal geometry. Using (5.13) with  $\Phi, f$  and  $e^\vee_a$  replaced by  $\phi, \varphi$  and  $e^{-\varphi} e^\vee_a$  and (5.17) with  $f_1, f_2$  and  $e^\vee_a$  substituted by  $\varphi, 0$  and  $e^{-\varphi} e^\vee_a$ , respectively, one verifies that  $\delta_{ia} \iota(e^\varphi e_a) T_{ei}(\phi) = \mathcal{C}_R(0, e^{-\varphi} e^\vee)$ .  $\mathcal{C}_R(0, e^{-\varphi} e^\vee) = 0$ , by (5.6), because, by (3.2), the local background  $e^{-\varphi} e^\vee_a$  is flat. So,  $\delta_{ia} \iota(e^\varphi e_a) T_{ei}(\phi) = 0$ . This relation yields (5.10) immediately upon using (5.9) and recalling (3.1) and (3.5). Finally, from (5.2) and (5.4), we obtain the symmetry relation  $\delta_{ia} T_{ei}(0) \wedge e^{-\varphi} e^\vee_a = 0$ . From here, (5.11) follows upon using (5.9) and recalling (3.2) and (3.7). Next, by using the symmetry relation (5.2) and the Ward identity (5.3) and exploiting relations (3.9) and (3.17), one has

$$\begin{aligned} d \star \left[ e^{3\varphi} \mathcal{T}_{ea}(0, e^\vee) \right] &= e^\varphi \left[ d\varphi \wedge \star \mathcal{T}_{ea}(0, e^\vee) + d \star \mathcal{T}_{ea}(0, e^\vee) \right] \\ &= e^\varphi \langle \mathcal{T}_{eb}(0, e^\vee), e_b \rangle d\varphi \wedge \star e^\vee_a. \end{aligned} \quad (5.20)$$

From the Ward identity (5.5) with  $f$  and  $e^\vee_a$  replaced by  $\varphi$  and  $e^{-\varphi} e^\vee_a$ , one deduces further that

$$d \star \mathcal{L}_{Ra}(\varphi, e^{-\varphi} e^\vee) = \mathcal{C}_R(\varphi, e^{-\varphi} e^\vee) d\varphi \wedge \star e^{-\varphi} e^\vee_a. \quad (5.21)$$

In deriving this relation, one uses that  $d \star (e^{-\varphi} e^\vee_a) = 0$ , by (3.2). Now, by (5.19),  $d \star T_{ei}(0)$  is given by the difference of the left hand sides of eqs. (5.20) and (5.21), which vanishes

by (5.13) with  $\Phi$ ,  $f$  and  $e^\vee_a$  replaced by 0,  $\varphi$  and  $e^{-\varphi}e^\vee_a$  and by (3.2) and (3.9). Hence,  $d \star T_{ei}(0) = 0$ . From here, using (5.9), (5.12) follows.  $QED$

The above treatment is essentially a reformulation of the classic results of ref. [18] highlighting the connection with Kulkarni geometry.

As noticed earlier,  $T_e$  does not transform as its classical counterpart under coordinate changes. In fact, on  $U_\alpha \cap U_\beta \neq \emptyset$ , one has

$$T_{e\alpha} = \zeta_{3\alpha\beta}(T_{e\beta} + \varrho_{\alpha\beta}), \quad (5.22)$$

where  $\zeta_3$  is defined in (2.34) and

$$\varrho_{\alpha\beta} = \frac{1}{4} \left[ \mathcal{L}_{R\beta 0}(dx_\beta, \ln(|\eta^+_{\alpha\beta}|/|\eta^-_{\alpha\beta}|)) - \mathcal{L}_{R\beta e}(dx_\beta, \ln(|\eta^+_{\alpha\beta}|/|\eta^-_{\alpha\beta}|))j_e \right]. \quad (5.23)$$

*Proof.* Set  $t_{\alpha\beta} = \ln(|\eta^+_{\alpha\beta}|/|\eta^-_{\alpha\beta}|)$ . Then,

$$\begin{aligned} \mathcal{T}_{R\alpha a}(\varphi_\alpha, e^{-\varphi_\alpha}e^\vee_\alpha) &= e^{3\varphi_\alpha} \left[ \mathcal{T}_{R\alpha a}(0, e^\vee_\alpha) - \mathcal{T}_{R\alpha a}(-\varphi_\alpha, e^\vee_\alpha) \right. \\ &\quad \left. - \frac{1}{4}\mathcal{C}_R(-\varphi_\alpha, e^\vee_\alpha)e^\vee_{\alpha a} \right] \\ &= e^{3\varphi_\beta - 3t_{\alpha\beta}} r_{\alpha\beta ab} \left[ \mathcal{T}_{R\beta b}(0, e^\vee_\beta) - \mathcal{T}_{R\beta b}(-\varphi_\beta + t_{\alpha\beta}, e^\vee_\beta) \right. \\ &\quad \left. - \frac{1}{4}\mathcal{C}_R(-\varphi_\beta + t_{\alpha\beta}, e^\vee_\beta)e^\vee_{\beta b} \right] \\ &= e^{3\varphi_\beta - 3t_{\alpha\beta}} r_{\alpha\beta ab} \left[ e^{-3\varphi_\beta} \mathcal{T}_{R\beta b}(\varphi_\beta, e^{-\varphi_\beta}e^\vee_\beta) + \mathcal{T}_{R\beta b}(-\varphi_\beta, e^\vee_\beta) \right. \\ &\quad \left. + \frac{1}{4}\mathcal{C}_R(-\varphi_\beta, e^\vee_\beta)e^\vee_{\beta b} - \mathcal{T}_{R\beta b}(-\varphi_\beta + t_{\alpha\beta}, e^\vee_\beta) \right. \\ &\quad \left. - \frac{1}{4}\mathcal{C}_R(-\varphi_\beta + t_{\alpha\beta}, e^\vee_\beta)e^\vee_{\beta b} \right] \\ &= e^{-3t_{\alpha\beta}} r_{\alpha\beta ab} \left[ \mathcal{T}_{R\beta b}(\varphi_\beta, e^{-\varphi_\beta}e^\vee_\beta) - \mathcal{T}_{R\beta b}(t_{\alpha\beta}, e^{-\varphi_\beta}e^\vee_\beta) \right. \\ &\quad \left. - \frac{1}{4}\mathcal{C}_R(t_{\alpha\beta}, e^{-\varphi_\beta}e^\vee_\beta)e^{-\varphi_\beta}e^\vee_{\beta b} \right], \end{aligned} \quad (5.24)$$

where  $r_{\alpha\beta}$  is the same  $SO(4)$  valued function as that appearing in (3.38). Here, the first identity is proven by applying (5.18) with  $f_1$ ,  $f_2$  and  $e^\vee_a$  substituted by  $\varphi_\alpha$ ,  $-\varphi_\alpha$  and  $e^\vee_{\alpha a}$ , respectively. The second identity follows from (3.29), (3.38) and the relation

$$\mathcal{T}_{R\alpha a} = r_{\alpha\beta ab} \mathcal{T}_{R\beta b} \quad (5.25)$$

analogous to (4.14). The third identity is proven by applying (5.18) with  $f_1$ ,  $f_2$  and  $e^\vee_a$  substituted by  $\varphi_\beta$ ,  $-\varphi_\beta$  and  $e^\vee_{\beta a}$ , respectively. The fourth and final identity is shown



by applying (5.17) and (5.18) with  $f_1$ ,  $f_2$  and  $e^\vee_a$  substituted by  $t_{\alpha\beta}$ ,  $-\varphi_\beta$  and  $e^\vee_{\beta a}$ , respectively. Next, one has

$$\begin{aligned}\mathcal{C}_R(\varphi_\alpha, e^{-\varphi_\alpha} e^\vee_\alpha) e^{-\varphi_\alpha} e^\vee_{\alpha a} &= e^{3\varphi_\alpha} \mathcal{C}_R(0, e^\vee_\alpha) e^\vee_{\alpha a} \\ &= e^{3\varphi_\beta - 3t_{\alpha\beta}} r_{\alpha\beta ab} \mathcal{C}_R(0, e^\vee_\beta) e^\vee_{\beta b} \\ &= e^{-3t_{\alpha\beta}} r_{\alpha\beta ab} \mathcal{C}_R(\varphi_\beta, e^{-\varphi_\beta} e^\vee_\beta) e^{-\varphi_\beta} e^\vee_{\beta b}.\end{aligned}\tag{5.26}$$

The first identity is obtained by applying (5.18) with  $f_1$ ,  $f_2$  and  $e^\vee_a$  substituted by  $\varphi_\alpha$ ,  $-\varphi_\alpha$  and  $e^\vee_{\alpha a}$ , respectively. The second identity follows from (3.29) and (3.38). The third identity is proven by applying (5.17) with  $f_1$ ,  $f_2$  and  $e^\vee_a$  substituted by  $\varphi_\beta$ ,  $-\varphi_\beta$  and  $e^\vee_{\beta a}$ , respectively. Combining (3.27), (3.29), (3.39), (5.24) and (5.26) with (5.8) and (5.19) and recalling (2.34), one checks easily that the matching relation of the  $T_{R\alpha}$  is given by (5.22)–(5.23). *QED*

The compatibility of (5.22) and (5.10)–(5.12) entails the following relations

$$\text{Re}(\varrho_{\alpha\beta} \iota(\partial_{\bar{q}\beta})) = 0, \tag{5.27}$$

$$\text{Re}(dq_\beta \wedge \varrho_{\alpha\beta}) = 0, \tag{5.28}$$

$$d \star_\beta \varrho_{\alpha\beta} = 0. \tag{5.29}$$

*Proof.* The verification of (5.27) and (5.28) is completely straightforward. To show (5.29), one has take into account the fact that, if local quaternionic 1-forms  $\nu_\alpha$  satisfy  $\text{Re}(\nu_\alpha \iota(\partial_{\bar{q}\alpha})) = 0$  and  $\text{Re}(dq_\alpha \wedge \nu_\alpha) = 0$ , then the equation  $d \star_\alpha \nu_\alpha = 0$  is covariant under the matching relation  $\nu_\alpha = \zeta_{3\alpha\beta} \nu_\beta$ . *QED*

From (5.23), it appears that  $\varrho_{\alpha\beta}$  depends only on the underlying conformal geometry. So, the matching relation (5.22) is completely analogous to that of the conformally invariant energy–momentum tensor in 2–dimensional conformal field theory and  $\varrho_{\alpha\beta}$  is a 4–dimensional generalization of the Schwarzian derivative.

The form of the conformal anomaly [19–20] is determined up to a term of the form  $\delta\mathcal{K}(e^\vee)$ , where  $\delta$  denotes variation with respect to the scale of  $e^\vee_a$  and  $\mathcal{K}(e^\vee)$  is a local functional of  $e^\vee_a$ . The form of the anomaly can be rendered simpler by means of a convenient choice of  $\mathcal{K}$ . A further simplification is yielded by the local conformal flatness of the background  $e^\vee_a$  of eq. (3.2), which makes the contribution containing the square of the Weyl tensor vanish identically. In this way the conformal anomaly can be cast as

$$\delta\mathcal{I}_e = \frac{\kappa}{128\pi^2} \int_X \left[ 32\pi^2 \epsilon - \frac{2}{3} d \star ds \right] \langle \delta e^\vee_a, e_a \rangle, \tag{5.30}$$

where  $\epsilon$  is the Euler density, defined above (3.25), and  $s$  is the Ricci scalar.  $\kappa$  is a real coefficient called central charge. In fact, the expression of the anomaly is simpler than it looks at first glance. A detailed calculation, exploiting the local conformal flatness of  $e^\vee_a$ , shows that it can be written in the form

$$\delta\mathcal{I}_e = \frac{32\kappa}{\pi^2} \int_X \square \star \square \varphi \delta\varphi, \quad (5.31)$$

where  $\square = \frac{1}{16}d \star d$  is the D'Alembert operator. In this form, the similarity with the standard 2-dimensional case is apparent. As a byproduct, we learn also that  $\square \star \square \varphi$  belongs to  $\Omega^4(X)$ , an interesting geometric result.

The Riegert action corresponding to the anomaly (5.19) is given by [16–17]

$$\begin{aligned} \mathcal{I}_R(f, e^\vee) = \frac{\kappa}{16\pi^2} \int_X \left[ d \star df \wedge \star d \star df - \frac{2}{3} s df \wedge \star df \right. \\ \left. + 2e_a(f) S_a \wedge \star df + \left( 16\pi^2 \epsilon - \frac{1}{3} d \star ds \right) f \right]. \end{aligned} \quad (5.32)$$

In the locally conformally flat background  $e^\vee_a$  of eq. (3.2),  $\mathcal{I}_R$  can be written as

$$\mathcal{I}_R(f, \varphi) = \frac{32\kappa}{\pi^2} \int_X \left[ \frac{1}{2} f \square \star \square f + \square \star \square \varphi f \right]. \quad (5.33)$$

When written in this form, the resemblance of the 4-dimensional Riegert action and the well-known 2-dimensional Liouville action is striking. The calculation shows also that  $\square \star \square$  is a globally defined differential operator of order 4 mapping  $\Omega^0(X)$  into  $\Omega^4(X)$ <sup>7</sup>.

It is now straightforward though quite tedious to compute  $T_e$ . Set

$$\begin{aligned} P(f) = 4df \partial_q \star \square f + 4\partial_q f d \star \square f - \frac{1}{12} d \partial_q \star (df \wedge \star df) \\ + \frac{1}{2} dx^i \star (d \partial_{x^i} f \wedge \star d \partial_q f) - 8d \partial_q f \star \square f - \frac{4}{3} d \partial_q \star \square f \\ + d\bar{q} \left[ 8(\star \square f)^2 - \frac{1}{6} \star \square \star (df \wedge \star df) + \frac{16}{3} \star \square \star \square f \right]. \end{aligned} \quad (5.34)$$

Then,  $T_e(\phi)$  is given by

$$T_e(\phi) = e^{3\varphi} \mathcal{T}_e(e^{-\varphi\Lambda}\phi, e^\vee) - \kappa P(\varphi), \quad (5.35)$$

---

<sup>7</sup> This operator, as many others, could have been included in the list of the natural differential operators of a Kulkarni 4-fold studied in section 2. To keep the size of this paper reasonable, we decided to limit our discussion to  $\square$  and  $\bar{\partial}_{R,L}$ .

where  $\mathcal{T}_e = \frac{1}{4}(\mathcal{T}_{e0} - \mathcal{T}_{ee}j_e)$ . The 4-dimensional Schwarzian derivative  $\varrho_{\alpha\beta}$  defined in (5.23) is given explicitly by

$$\varrho_{\alpha\beta} = \kappa P_\beta(\ln(|\eta^+_{\alpha\beta}|/|\eta^-_{\alpha\beta}|)). \quad (5.36)$$

#### *The operator product expansions*

We shall now analyze the structure of the operator product expansions for the simple free models studied in section 4.

Consider the complex boson  $\Phi$  described by the action (4.19). The quantum theory is best formulated in terms of the conformally invariant field  $\phi$  governed by the action (4.21). Inside normalized conformally invariant quantum correlators, the classical field equations (4.23) hold up to contact terms

$$\square\phi = 0 \quad \text{up to contact terms.} \quad (5.37)$$

Hence, the correlators are harmonic in the insertion points of the field  $\phi$  and its complex conjugate, provided such points remains distinct. Since a real harmonic function can be expressed as the real part of a Fueter holomorphic function [7], Fueter analyticity is relevant in this model. From the form of the action (4.21), it follows in particular that

$$\begin{aligned} -\frac{8}{\pi^2}\phi(q_2)\square_1\bar{\phi}_c(q_1) &= \delta^4(q_2 - q_1) \star 1_2, \\ -\frac{8}{\pi^2}\square_2\phi(q_2)\bar{\phi}_c(q_1) &= \delta^4(q_2 - q_1) \star 1_1. \end{aligned} \quad (5.38)$$

This relation can be easily integrated on a given coordinate patch, yielding

$$\phi(q_2)\bar{\phi}_c(q_1) = \frac{1}{2|q_2 - q_1|^2} + \text{regular harmonic terms.} \quad (5.39)$$

*Proof.* From distribution theory, one can show easily that  $\partial_{\bar{q}}\partial_q|q - q_0|^{-2} = -\frac{\pi^2}{4}\delta^4(q - q_0)$  in  $\mathcal{D}'(\mathbb{H})$ . Further, it is known [7] that there is no singular harmonic function less singular than  $|q - q_0|^{-2}$ . *QED*

Consider the Dirac fermion  $\Psi$  described by the action (4.28). It is more convenient to formulate the quantum theory in terms of the conformally invariant fields  $\psi^\pm$  governed by the action (4.35). (We assume  $|v_0| = 1$  here for the sake of simplicity). Inside normalized conformally invariant quantum correlators, the classical field equations (4.39) hold up to contact terms

$$\psi^+\bar{\partial}_R = 0, \quad \bar{\partial}_L\psi^- = 0 \quad \text{up to contact terms.} \quad (4.39)$$

Hence, the quantum correlators are right (left) Fueter holomorphic in the insertion points of the field  $\psi^+$  ( $\psi^-$ ), provided such points do not coincide. This statement must carry

a warning. Since the fields  $\psi^\pm$  and the Fueter operators  $\bar{\partial}_{R,L}$  are valued in the non commutative quaternion field, the statement holds provided  $\bar{\partial}_R$  ( $\bar{\partial}_L$ ) acts on  $\psi^+$  ( $\psi^-$ ) within the correlators. The above shows the relevance of Fueter analyticity in the present fermionic model. From the form of the action (4.35), it follows in particular that

$$\begin{aligned}\frac{2}{\pi^2}\psi^-(q_2)\tilde{\psi}^+(q_1)\bar{\partial}_{R1} &= \delta^4(q_2 - q_1)d\bar{q}_2, \\ \frac{2}{\pi^2}\bar{\partial}_{L2}\psi^-(q_2)\tilde{\psi}^+(q_1) &= \delta^4(q_2 - q_1)d\bar{q}_1.\end{aligned}\tag{5.41}$$

This relation can be integrated on any given coordinate patch, producing

$$\psi^-(q_2)\tilde{\psi}^+(q_1) = \frac{\bar{q}_2 - \bar{q}_1}{|q_2 - q_1|^4} + \text{terms right (left) Fueter holomorphic in } q_1 \text{ (} q_2 \text{)}.\tag{5.42}$$

*Proof.* From distribution theory, one knows that  $[(\bar{q} - \bar{q}_0)|q - q_0|^{-4}]\partial_{\bar{q}R} = \partial_{\bar{q}L}[(\bar{q} - \bar{q}_0)|q - q_0|^{-4}] = \frac{\pi^2}{2}\delta^4(q - q_0)$  in  $\mathcal{D}'(\mathbb{H})$ . Further, it is known [7] that there is no singular left/right Fueter analytic function that is less singular than  $(\bar{q} - \bar{q}_0)|q - q_0|^{-4}$ . *QED*

The above analysis shows that Fueter analyticity provides useful information on the structure of the operator product expansions of free fields. It remains to be seen if this will be of any help in computations.

## 6. Conclusions and outlook

In the first part of this paper, we have tried to formulate the theory of Kulkarni 4-folds in a way that parallels as much as possible the customary formulation of the theory of Riemann surfaces, highlighting in this way their analogies. This has been possible thanks to the existence of an integrable quaternionic structure and of an associated natural notion of analyticity, Fueter analyticity. We have also seen that a Kulkarni 4-folds is equipped with a canonical conformal equivalence class of locally conformally flat metrics and that the Riemannian geometry of such metrics is particularly simple.

In the second part of the paper, we have argued that Kulkarni geometry is the natural geometry of 4-dimensional conformal field theory by showing that the action functional, the field equations, the energy-momentum tensor and its Ward identity and the operator product expansions take a simple form for a conformal field theory on a Kulkarni 4-fold.

We have not analyzed yet the implications of the geometric setting on the operator product expansion of the energy-momentum tensor. This matter is left for future work [21]. We believe in fact that the customary energy-momentum tensor, describing the response of the system to an arbitrary variation of an arbitrary background metric,

might not be the relevant geometric field. One should consider instead a modified energy–momentum tensor representing the response of the system to an arbitrary variation of an arbitrary locally conformally flat background metric preserving local conformal flatness. This would be the true analogue of the energy–momentum tensor of 2–dimensional conformal field theory, as for a 4–fold admitting locally conformally flat metrics, unlike for a 2–fold, not all metrics are automatically locally conformally flat. One may speculate that the improved energy–momentum tensor just described might obey operator product expansion of universal form as in 2 dimensions. This remains to be seen. In any case, to carry out the above project requires the elaboration of the Kulkarni analogue of the Beltrami parametrization of conformal structures, a major mathematical task in itself with ramifications also in geometry.

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